Multiple Weighted Expected Utility Theory

by

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ABSTRACT

Chapter 1: Expected Utility without the Independence and Completeness Axioms

This paper establishes an intuitive foundation for modeling preferences which do not obey either completeness or independence. From the existing literature, we know that preferences which violate the independence axiom, as demonstrated by the famous Allais paradox, are represented by a utility function that is not linear in the probabilities, and hence individual components cannot be freely substituted for equivalent ones. On the other hand, incomplete preferences can only be represented by a set of utility functions, and whenever they disagree on how to rank a pair of lotteries, the decision maker can exhibit neither preference nor indifference, and must judge the alternatives to be incomparable. Therefore, preferences that violate both of these axioms simultaneously must have a utility representation that exhibits both multiplicity and non-linearity, reflecting an internal consensus among a set of decision criteria, none of which admit the strong substitution property implied by the independence axiom.

Chapter 2: Weighted Expected Utility without the Completeness Axiom

This paper axiomatizes a utility representation for incomplete preferences that violate independence and satisfy only a weaker ratio substitution property. I show that such a representation is given by a set of weighted linear utility functions, each of which generates an indifference map consisting of a set of projectively parallel indifference curves originating from a source point outside of the simplex of lotteries. The overall indifference map is therefore constructed by superimposing the maps corresponding to the individual decision criteria, with the locations of the source points determining the decision maker's sets of utility and weight functions. These respectively represent a range of conflicting tastes and a set of disparate perceptions of mixture distorting the evaluation of subjective probabilities, and incompleteness may arise from any disagreement in either or both.

Chapter 3: Incomplete Preferences with Conflicting Tastes and Perceptions

This paper considers variations of the general model where the multiplicity of decision criteria is restricted to either the utility or weight functions alone. I find that these cases differ fundamentally in how the level of indecision exhibited by the decision maker varies with her point of reference, standing in contrast to standard multi-utility models where the degree of indecision must remain constant throughout the space of lotteries. In the multiple utility, single weight case the decision maker is unsure only of her tastes, so that her inability to rank alternatives derives purely from her incapacity to evaluate the component outcomes and is hence mitigated by mixture of prospects. On the other hand, in the single utility, multiple weight case the decision criteria disagree only on perception of risk, so that incompleteness is instead exacerbated by mixture. This allows us to discern the composition of an individual's internal decision making process from patterns of observed choice or lack thereof.

Primary Reader - Edi Karni

Secondary Reader - M. Ali Khan

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1. EXPECTED UTILITY WITHOUT THE INDEPENDENCE AND COMPLETENESS AXIOMS

1.1 Introduction

The purpose of this dissertation is to build a decision theoretic model of choice under risk for preferences that may simultaneously violate both the completeness and independence axioms. As respectively the first and last of the assumptions made by von Neumann and Morgenstern (1947) [24] their seminal expected utility theory, these properties play a critical role in giving the model its intuitive simplicity. In the years since however, both have come under repeated scrutiny, as their supposed status as fundamental tenets of rationality have been questioned on both normative and descriptive grounds, and a dazzling variety of theories have been developed to accommodate patterns of choice behavior that do not adhere to them.

Up to this point however, research has been split into distinct lines, focused on modeling preferences without completeness while leaving the independence axiom intact, or vice versa. Our objective is to weave these separate strands in the literature into a single coherent framework. This opening chapter will review the development and evolution of models of decision making in the presence of risk, as well as consider how the ideas put forth in the literature can be combined into a unified model, and examine the distinct challenges and difficulties that this task presents.

1.1.1 Relaxing Completeness

It was in fact von Neumann and Morgenstern themselves who first disputed the notion that an individual presented with a pair of alternatives should always be able to indicate a preference for at least one of them, and they postulated that relaxing this assumption would lead to a "many-dimensional vector concept of utility." Hence, from the very beginning the concepts of incomplete preferences and multi-utility representations have been fundamentally entwined. Indeed, perhaps the basic flaw inherent to the completeness axiom is the conflation of the colloquial meaning of indifference with its formal definition in a decision theoretic context, which imparts an additional notion of equivalence that has proven somewhat troublesome to disentangle. For example, many people would likely describe themselves as "indifferent" between Coke and Pepsi, in the sense that they would give little consideration to which of the two they would prefer to drink if given a choice. However, few would adhere to this attitude in the stronger sense implied by expected utility, in which any slight alteration to the ingredients would decisively tip the scales in favor of one or the other. Hence, the relevant attitude of decision makers here might be more accurately termed as indecisiveness or incomparability, notions which lie outside the scope of standard expected utility theory, by virtue of its reliance on the completeness axiom.

The general concept of incomplete preferences encompasses a wide variety of phenomena. As in the example of Luce (1956) [19] regarding a decision maker's indifference towards the addition a single granule of sugar to a cup of coffee, indecision may arise if the distinction between her alternatives is too insignificant to attract her notice, even though she may express strict preference when this disparity is magnified to a sufficient degree. On the other hand, it is equally plausible that when two prospects differ so much that any comparison between them would be absurd, such as in a choice between a pet snake and a lifetime supply of mechanical pencil lead, the decision maker must be indecisive by rule as whatever she chooses would not reveal useful information about her underlying preferences anyway. Furthermore, she may simply lack sufficient information or understanding about the alternatives themselves. In each of these instances, indecision does not necessarily cripple the agent to such a degree that she becomes completely unable to choose if forced to do so, but the observed patterns of decisions in such instances will likely be fickle or inconsistent, and reveal little about her underlying tastes.

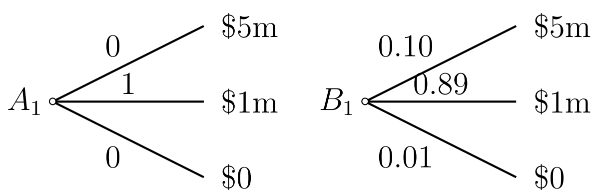
As Aumann (1962) [2] astutely observes, the real numbers are completely ordered and hence no single real-valued utility function can adequately represent a partially ordered preference relation. He shows however that there exist multiple utility functions compatible with the preference ordering, in the sense that if one lottery is strictly preferred to another then it must be assigned a higher value. The converse does not hold for any such utility however, as preferences would be complete if it did, so that none of these functions alone represents the preference relation and hence they are not unique in the sense of being positive affine transformations of one another. Therefore, an individual with partially ordered preferences may be interpreted as having several decision criteria representing a set of objective functions, and expresses strict preference if and only if each of these agrees on a ranking of two alternatives. Any disagreement between conflicting criteria will manifest as incomparability, distinct from indifference which would instead require each utility to value both alternatives equally. As the latter is exceedingly unlikely unless all of the utilities are identical, this gives some indication that indecisive behavior overwhelmingly reflects incompleteness rather than equivalence. Given this characterization of incomplete preferences as the product of an internal "committee," we can easily conceive of such a relation describing a multi-person preference as in Baucells and Shapley (1998) [3]. In their model, an incomplete relation is taken as the joint preference of a coalition of agents, each of whose individual preferences may in turn be incomplete.

All social choice problems can therefore also be understood within the general context of incomplete preferences. Any mechanism chosen to resolve conflicts among the decision criteria, whether internal or external, must yield patterns of observed choices which do not always obey the same axioms that decisive actions conducted by the same agent or coalition must follow. Bewley (1986) [4] proposed an inertia assumption, where a decision maker switches her behavior only when presented with an alternative that strictly dominates her existing program. For example, consider the problem of participation in an organ donor program, in which it is commonly observed that regardless of whether an opt-in or opt-out program is employed, people tend to stick with their initial assignment even though switching is relatively costless. However tempting it may seem at this point to simply dismiss any and all seemingly irrational behavior as merely the result of haphazard resolution of incompleteness, we must still account for the fact that preferences cannot simply be empty and that decisive behavior not only exists but may itself fail to adhere to the other treasured assumptions of expected utility.

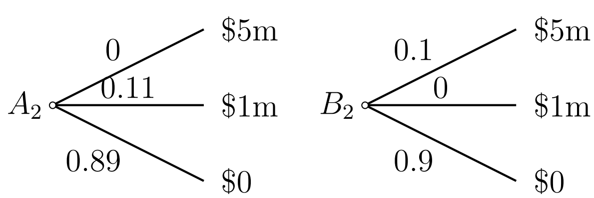
1.1.2 Relaxing Independence

As suggested by Dubra and Ok (2002) [11], agents may attempt to expand their set of comparable pairs by devising a procedure that applies certain principles of rationality, centrally the independence axiom. As its name suggests, this assumption implies that decision makers should not treat parts of a whole which do not interact as if they do, so that an ordering of two alternatives is not reversed if both are mixed in equal proportion with a third. Under completeness, it ensures that the utility can be expressed as the expectation of the value obtained from the outcome of a lottery and hence will be linear in the probabilities, and if completeness is relaxed, it allows any function within the set of compatible utilities to be decomposed in the same manner. Thus the independence axiom plays a critical role not only in imposing structure on choice behavior, but also in generating it from a relatively sparse set of initial rankings by taking advantage of the substitution property it provides. For example, an individual who knows that she prefers a cup of coffee to a cup of tea may use the independence axiom to deduce that she prefers any lottery that gives coffee with some positive probability to an otherwise identical lottery that instead gives tea with the same probability.

Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13] devise multiple expected utility models where incomplete preferences that obey independence are represented by a set of linear utility functions. Furthermore, that the set of utilities is unique up to the closure of the convex cone they generate, mirroring the result in expected utility of uniqueness up to a positive affine transformation. However, as these conclusions depend on the independence axiom, they remain susceptible to the same pitfalls as any model relying on the principle of linearity in probability, most famously demonstrated in the paradox of Allais (1953) [1].



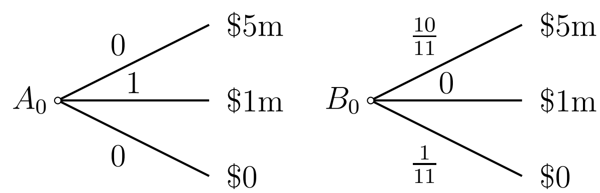
(a) $A_1 \succ B_1$



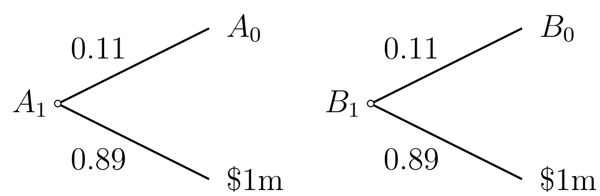
(b) $A_2 \prec B_2$

Fig. 1.1: Allais Paradox - Common Consequence Effect

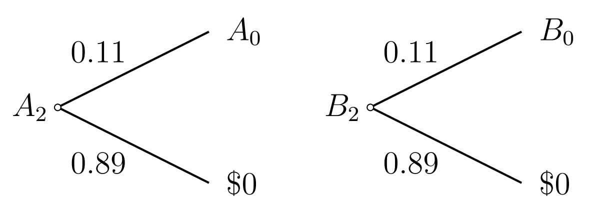
Figure 1.1 demonstrates the “common consequence effect” of the Allais paradox, with each tree representing a gamble that gives each of the prizes of $0, $1 million, or $5 million with the indicated probability, and a decision maker facing, for $i = 1, 2$ a choice between A_i and B_i. In the first instance, Allais postulated that people typically prefer the certain outcome, judging a marginal gain in expected payoff to not be worth the risk of getting nothing, and set $A_1 \succ B_1$. On the other hand, in the second instance they would likely perceive the chance of receiving a positive payoff to be low in any event and thus go for the higher potential gain, so that $A_2 \prec B_2$.



(a) $A_0(\succ / \prec)B_0$



(b) $0.11A_0 + 0.89(\$1m) \succ 0.11B_0 + 0.89(\$1m)$

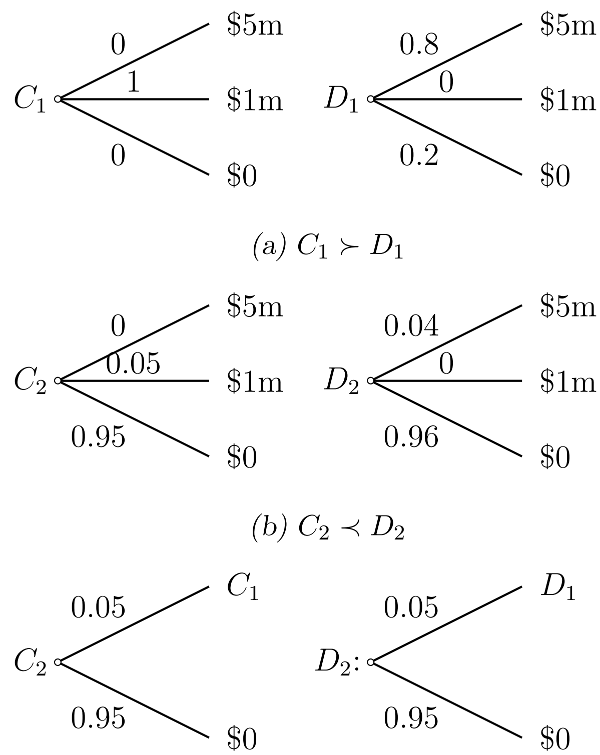


(c) $0.11A_0 + 0.89(\$0) \prec 0.11B_0 + 0.89(\$0)$

Fig. 1.2: Common Consequence Effect Expressed with Compound Lotteries

However, these preferences are inconsistent with the independence axiom as they fail to separate out the titular common consequence between each A_i and B_i that gives with probability 0.89 a payoff of \$1 million for $i = 1$ and \$0 for $i = 2$, which becomes clear if they are expressed as compound lotteries as in Figure 1.2. The separability imparted by the independence axiom would prescribe that the decision maker rank these pairs only according to their differences, reflected in a choice between A_0 and

B_0, so that she judges A_i to be either better or worse than B_i for both $i = 1, 2$, and cannot reverse her preference if the consequence in the lower branch of the first chance node is changed. This shows that decision makers do not exhibit the same degree of risk aversion in every circumstance and hence are unable to separate lotteries into their components as the independence axiom would prescribe.



(a) $C_1 \succ D_1$

(b) $C_2 \prec D_2$

(c) $0.05C_1 + 0.95(\$0) \prec 0.05D_1 + 0.95(\$0)$

Fig. 1.3: Allais Paradox - Common Ratio Effect

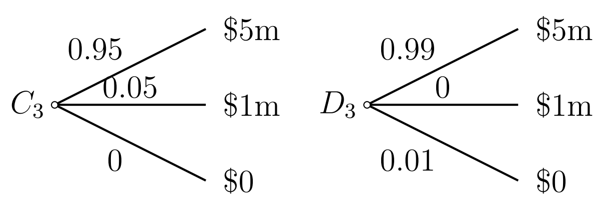
Figure 1.3 demonstrates the related "common ratio effect," where individuals are commonly observed following the same logic as in the previous example, setting $C_1 \succ D_1$ and $C_2 \prec D_2$. Once again, expressing C_2 and D_2 as compound lotteries reveals that the choice in the second scenario should be consistent with that of the first since the ratios between the chances of winning non-zero prizes in C_i and D_i are identical for $i = 1, 2$. Experimental evidence from MacCrimmon and Larsson (1979) [21] and Kahneman and Tversky (1979) [16] confirms that both of these paradoxes hold under a range of values for the various payoffs and probabilities. This indicates that independence is systematically violated in practice and furthermore that these violations are the product of decisive action, and not simply incoherent choices made between fundamentally incomparable pairs.

The problem of resolving this paradox depends on the interpretation of what exactly it is supposed to represent. In the prospect theory of Kahneman and Tversky (1979) [16], violations of independence arise because decision makers are unable to perceive objective probabilities properly, and instead have a distorted perception of risk given by a decision weight function. In the common ratio Allais paradox for example, the preference for C_1 over D_1 may be understood as representing the agent's true degree of risk aversion based on her valuation of the prizes themselves, whereas that of D_2 over C_2 merely indicates an inability to see any difference between probabilities of 0.05 and 0.04. When evaluating these prospects, the decision maker only perceives that both offer only a "small" chance of winning, and naturally the one that offers a higher potential payoff is better, even though the independence axiom would prescribe otherwise. This reflects a decision weight function $w(\cdot)$ that is distorted near both extremes, so that anything within the neighborhood of either certainty or impossibility is perceived nearly identically. However, since the decision weights are applied without regard to the overall probability distribution that defines the prospect, the possibility of over-weighting multiple low probability events can lead to violations of first-order stochastic dominance. For example, if $w(0.06) + w(0.04) > w(0.1)$, then a prospect that pays \$100 with probability 0.1 would be ranked strictly worse than one that paid \$99 with probability 0.06 and \$98 with probability 0.04 even though the former strictly dominates the latter. The effect of such a distortion is more properly captured by applying it only to the perception of extreme events, such as in the cumulative prospect theory of Kahneman and Tversky (1992) [17] which incorporates the rank-dependent cumulative weight function devised in the anticipated utility theory of

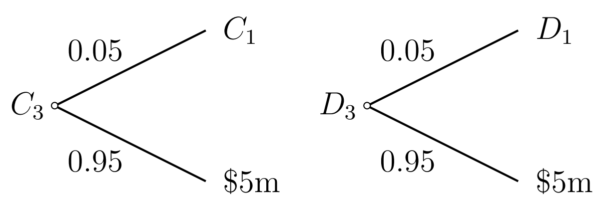
Quiggin (1982) [23].

However, any notion of a rank-dependent utility necessarily depends on the ability to actually rank each of the outcomes, which makes adapting such models to account for potentially incomplete preferences a difficult proposition. We instead consider a class of models that replace the independence axiom with the weaker betweenness property, with an eye towards applying it to pairs of incomparable alternatives in addition. That is to say, if the decision maker is unable to rank two lotteries, then she must also find any pair of mixtures of these lotteries to be incomparable as well. Fishburn (1983) [12] showed that under completeness, replacing independence with betweenness yields a representation by a continuous and increasing utility function that is not necessarily linear, and Dekel (1986) [8] further shows that this utility is given by the solution to an implicit function. Chew and MacCrimmon (1979) [7] further impose a weakened form of substitution property implied by independence and obtain a utility that is a weighted expected value, with the weight function being applied to the outcomes, rather than to the probabilities as in prospect theory. This weight function can thus be taken as the degree of importance attached to each prize, which is determined separately both from the value that it would be assigned under certainty and the likelihood of receiving it, and represents a transformation of the space of the lotteries that skews the perception of risk toward the more heavily weighted prizes.

For example, a decision maker may be indifferent between a smartphone and $300 in cash but assign greater weight to the former, thus being more willing to enter a raffle that offers the phone over an otherwise identical one that gives a cash prize with equal probability, regardless of whether this probability is high or low. This would explain why tangible prizes are typically used rather than cash as incentives to enter contests or sign up for services. Since the value assigned to either prize must be identical since they are deemed equivalent in the absence of risk, and the probabilities of winning are also equal, this divergence from expected utility can be explained only by introducing a third component in the form of a weight function applied to the outcomes themselves. If the extreme outcomes weighted more heavily than middling ones, then the decision maker's indifference curves over the simplex of lotteries obey the "fanning out" property of Machina (1982) [22], exhibiting increasing risk aversion as prospects improve in the sense of first-order stochastic dominance. In the context of the Allais paradox, both the common consequence and common ratio paradoxes demonstrate this effect as the second pair of alternatives in each example are strictly worse than their counterparts in the first pair, and the experimental results demonstrate that subjects are accordingly more willing to take on risk in these situations. This reflects decision makers evaluating lotteries by a weighted expected value that assigns greater importance to the extreme payoffs of \$5 million and \$0 relative to the median outcome of \$1 million.



(a) $C_3(\succ / \prec)D_3$



(b) $0.05C_1 + 0.95(\$5m)(\succ / \prec)0.05D_1 + 0.95(\$5m)$

Fig. 1.4: "Reverse" Allais Paradox

It is somewhat less clear, however, whether this same conclusion holds when introducing risk makes both lotteries strictly better rather than worse, such as in Figure

1.4 where C_1 and D_1 are now instead mixed with the best possible prize. Weighted utility theory would prescribe that if $C_1 \succ D_1$ and $C_2 \prec D_2$, then we must necessarily have $C_3 \succ D_3$, as risk aversion must increase monotonically in terms of stochastic dominance. Gul (1991) [14] proposes an alternative theory of disappointment aversion that produces $C_3 \prec D_3$ instead, under the intuition that although there is a greater chance of receiving the worst possible prize in D_3, the overall probability of receiving a worse than expected outcome is higher in C_3. In this model, the simplex of lotteries is divided in two by the indifference curve intersecting the median outcome itself, so that indifference curves fan out in the lower half of the simplex of lotteries while symmetrically fanning in on the upper half. Chew (1989) [6] provides a general model that admits both possibilities, by weakening the substitution axiom further to allow asymmetry in this effect, so that the change in risk attitude depends on the particular outcome being mixed in, and may vary independently on the upper and lower halves.

1.1.3 Relaxing Both Axioms

We now consider the problem of introducing incompleteness into the class of betweenness-conforming preferences. Note that in the multi-utility models of Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13], the set of utility functions represent a range of potentially conflicting tastes that remain invariant throughout the space of lotteries, a direct consequence of the independence axiom. Furthermore, the degree of incompleteness, reflected in the level of disagreement among the utilities, is also fixed throughout, implying that the decision maker is equally indecisive everywhere. Naturally therefore, weakening both independence and completeness should combine the conclusions of both multi-utility and non-expected utility, yielding a set of non-linear utilities, which exhibit differing degrees of variation in tastes throughout the space of lotteries which consequently allows the level of indecision to vary as the criteria fall in and out of agreement with one another. Moreover, incompleteness may be generated within such a setting even in the absence of conflict in tastes. Yaari (1987) [25] developed a dual theory where utility was linear in payoff but not probability, so that risk aversion is generated not by attitude toward wealth but by perception, represented by a transformation function distorting the evaluation of objective probabilities much like the weight function in the model of Chew and MacCrimmon (1979) [7]. Maccheroni (2004) [20] showed that under incompleteness, this model admits a set of such distortion functions representing a range of such perceptions, so that just as under multi-utility, disagreement generates incomparability.

The objective of this dissertation is to devise a model of choice where preferences satisfy neither completeness nor independence, and are represented by a set utilities, each of which is linear in neither payments nor probabilities, representing multiple decision criteria that may reflect a range of either tastes, perceptions, or both, and whose level of disagreement may either grow or shrink in response to the introduction of risk. Such a model will incorporate all of the conclusions of various multi-utility and non-expected utility models within a single generalized framework, and admit a broad class of behavior under its umbrella. In this chapter, we investigate the various models described above, with a particular focus on how models that relax the completeness axiom rely on the independence axiom to obtain their representations, and vice versa. We consider how the conclusions obtained in both strands might fit within a single setting, and what assumptions are necessary and sufficient to generate the desired multiple weighted expected utility representation.

1.2 Setup

Let $\mathcal{C}$ denote a convex subset of a finite-dimensional linear space $\mathcal{L}$, so that it is a mixture set in the sense of Herstein and Milnor (1953) [15]. For reasons that will become clear shortly, we follow the example of Galaabaatar and Karni (2012) and take a strict preference relation $\succ \subseteq \mathcal{C} \times \mathcal{C}$ as the primitive. Assume that $\mathcal{C}$ is $\succ$-bounded, so that there exist maximal and minimal elements $\overline{p}$ and $\underline{p}$ respectively, such that $\overline{p} \succ p \succ \underline{p}$ for every $p \in \mathcal{C} \setminus \{\overline{p}, \underline{p}\}$. For the sake of brevity, for $\alpha \in [0,1]$ denote the proportional mixture of the best and worst elements by $\zeta_\alpha \equiv \alpha\overline{p} + (1-\alpha)\underline{p}$.

For example, we may consider the space of mixtures as a set of lotteries over some finite prize space X. Let $\mathcal{C} = \Delta(X)$, the space of all probability measures over X, so that every $p \in \Delta(X)$ is a lottery that gives each outcome $x \in X$ with probability $p(x)$, with $\sum_{x \in X} p(x) = 1$. For $x \in X$ denote by δ_x the degenerate lottery for which $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$, and for $p, q \in \Delta(X)$ and $\alpha \in [0,1]$, denote by $\alpha p + (1-\alpha)q \in \Delta(X)$ the mixture lottery for which $(\alpha p + (1-\alpha)q)(x) = \alpha p(x) + (1-\alpha)q(x)$ for every $x \in X$. The assumptions we introduce will ensure that the best and worst lotteries in this case will always be degenerate, so that there are $\overline{x}, \underline{x} \in X$ such that $\overline{p} = \delta_{\overline{x}}$ and $\underline{p} = \delta_{\underline{x}}$.

1.2.1 Expected Utility

Begin by enumerating the standard axioms of expected utility.

Axiom 1* (Weak Order) For every $p, q \in \mathcal{C}$, if $p \succ q$ then $\neg(p \prec q)$ and for every $p, q, r \in \mathcal{C}$, if $\neg(p \succ q)$ and $\neg(q \succ r)$ then $\neg(p \succ r)$.

Axiom 2* (Archimedean) For every $p, q, r \in \mathcal{C}$ if $p \succ r$ and $q \prec r$ then there are $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)q \succ r$ and $\beta p + (1-\beta)q \prec r$.

Axiom 3* (Independence) For every $p, q, r \in \mathcal{C}$ and $\alpha \in (0,1)$, $p \succ q$ if and only if $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$.

It is negative transitivity property in Axiom 1* to which we will refer as the completeness axiom, as it allows us to define a complete and transitive weak preference relation $\succeq$ as the negation of $\prec$. In the standard formulation, this relation is taken as the primitive with $\succ$ defined as its asymmetric part and an equivalence relation $\simeq$ as its symmetric part.

Definition (Weak Preference - EU) For every $p, q \in \mathcal{C}$, $p \succeq q$ if and only if $\neg(p \prec q)$.

Definition (Indifference - EU) For every $p, q \in \mathcal{C}$, $p \simeq q$ if and only if $p \succeq q$ and $p \preceq q$.

The structure of preferences under expected utility and all subsequent models is best understood by constructing one of the triangle diagrams popularized by Machina (1982) [22]. For every $p \in \mathcal{C} \setminus \{\overline{p}, \underline{p}\}$, let $\mathcal{L}(p)$ denote the plane spanned by the vectors $\overline{p}-p$ and $\underline{p}-p$. Thus every $\mathcal{L}(p)$ is the space of all mixtures that may be expressed as linear combinations of the three elements p, $\overline{p}$ and $\underline{p}$, so that for every $q \in \mathcal{L}(p)$ there are $\lambda, \theta \in \mathbb{R}$ such that $q = \lambda p+(1-\lambda)[\theta\overline{p}+(1-\theta)\underline{p}] = \lambda p+(1-\lambda)\zeta_\theta$, and $\mathcal{L}(q) = \mathcal{L}(p)$. In the case where $\mathcal{C} = \Delta(X)$ is the space of lotteries over three degenerate outcomes $X = \{x_1, x_2, x_3\}$ that are numbered in order of preference $\delta_{x_3} \succ \delta_{x_2} \succ \delta_{x_1}$, then we may depict the space of all lotteries over these outcomes in a triangle diagram by setting $p = \delta_{x_2}$, $\overline{p} = \delta_{x_3}$, and $\underline{p} = \delta_{x_1}$.

Figure 1.5 depicts the structure of preferences under the standard expected utility axioms, with each point in the triangle representing a unique mixture of the three elements. Under Axioms 1*-3*, the preference relation $\succ$ obeys the properties of mixture monotonicity and unique solvability.

Property (Mixture Monotonicity) For every $p, q \in \mathcal{C}$ and $\alpha, \beta \in (0, 1)$ such that α, β, $p \succ q$ if and only if $\alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q$.

Property (Unique Solvability) For every $p, q, r \in \mathcal{C}$ such that $p \succ r \succ q$, there is a unique $\alpha \in [0, 1]$ such that $r \simeq \alpha p + (1-\alpha)q$.

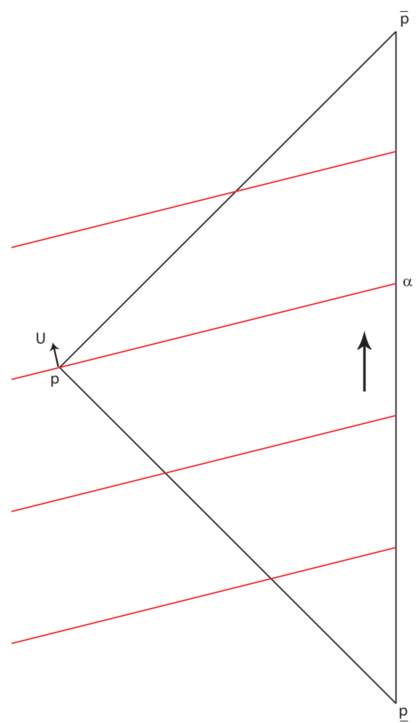


Fig. 1.5: Expected Utility

Hence for every p there is $\alpha \in [0,1]$ such that $p \simeq \beta p + (1-\beta)\zeta_\alpha \simeq \zeta_\alpha$ for every β, so that drawing the line connecting each p to its equivalent ζ_α shows that the diagram is composed of a set of linear indifference curves. Preference increasing in the upward direction defined by the vector $\overline{p} - \underline{p}$, with steeper lines representing greater risk aversion as the decision maker requires a higher probability of obtaining the best outcome before she is willing to gamble. The independence axiom further implies that all of the indifference curves must be parallel lines, and hence the risk attitude must remain constant throughout.

A ranking of the elements in $\mathcal{C}$ may thus be obtained by finding for each lottery the equivalent mixture ζ_α of the best and worst elements. The utility function $U : \mathcal{C} \mapsto \mathbb{R}$ is defined by setting $p \simeq \zeta_{U(p)}$ for every $p \in \mathcal{C}$, so that $p \succ q$ if and only if $U(p) > U(q)$. The independence axiom provides two important properties here that allow us to generate the representation from relatively sparse set of comparisons. Firstly, note that $U(p)$ defines $U(\cdot)$ over the entirety of $\mathcal{L}(p)$ since for every $q = \lambda p + (1-\lambda)\zeta_\theta \in$

$\mathcal{L}(p)$ we have by a single application of independence that $p \simeq \zeta_{U(p)}$ if and only if $\lambda p + (1-\lambda)\zeta_\theta \simeq \lambda\zeta_{U(p)} + (1-\lambda)\zeta_\theta = \zeta_{\lambda U(p)+(1-\lambda)\theta}$ and hence $U(q) = \lambda U(p) + (1-\lambda)\theta$. Note that by picking $\lambda, \theta \notin [0,1]$, we see that this utility function is defined not just in the simplex defined by $\{p, \overline{p}, \underline{p}\}$ but any lottery lying on the same plane. This ensures that we can define the representation over $\mathcal{L}(p)$ simply by defining a utility value for p itself, or equivalently doing the same for any other single mixture lying on $\mathcal{L}(p)$. Note that defining the utility in this manner must set $U(\overline{p}) = 1$ and $U(\underline{p}) = 0$, and thus is unique up to a positive linear transformation $\tilde{U}(p) = aU(p) + b$ for $a > 0$, arbitrarily setting $\tilde{U}(\overline{p}) = a + b$ and $\tilde{U}(\underline{p}) = b$.

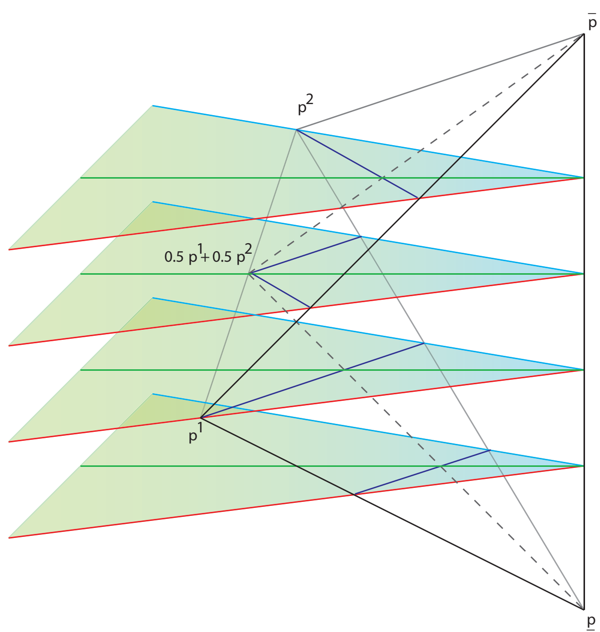


Fig. 1.6: Expected Utility in Three Dimensions

Secondly, for $p^1, p^2 \in \mathcal{C}$ such that $p^2 \notin \mathcal{L}(p^1)$, for $\pi \in [0,1]$ we have by two applications of independence that $\pi p^1 + (1-\pi)p^2 \simeq \pi\zeta_{U(p^1)} + (1-\pi)\zeta_{U(p^2)}$, and hence $U(\pi p^1 + (1-\pi)p^2) = \pi U(p^1) + (1-\pi)U(p^2)$. As demonstrated in Figure 1.6, this allows the triangle diagrams for $\mathcal{L}(p^1)$ and $\mathcal{L}(p^2)$ to be connected, joining the indifference curves in each to construct indifference planes which in turn define the triangle diagrams for every $\mathcal{L}(\pi p^1 + (1-\pi)p^2)$. Repeated applications of independence allow us to extend this result to higher dimensions, and the utility representation over the entirety of the mixture space to be constructed by taking some basis $\{p^1, \ldots, p^n\}$ of $\mathcal{C}$ and defining the utility over each $\mathcal{L}(p^i)$. For example, if $\mathcal{C} = \Delta(X)$, then we may take X itself as a basis, defining every lottery as $p = \sum_{x \in X} p(x)\delta_x$. Then by defining a utility function over outcomes $u : X \mapsto \mathbb{R}$ such that $\delta_x \simeq \zeta_{u(x)}$ for every $x \in X$, we have that $U(p) = \sum_{x \in X} p(x)u(x)$ for every $p \in \Delta(X)$, yielding the familiar representation.

$$p \succ q \Leftrightarrow \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)u(x)$$

1.2.2 Multiple Expected Utility

We now consider modifying the axioms to account for incompleteness, accomplished by weakening negative transitivity in Axiom 1* to transitivity.

Axiom 1 (Strict Partial Order) For every $p, q \in \mathcal{C}$, if $p \succ q$ then $\neg(p \prec q)$ and for every $p, q, r \in \mathcal{C}$, if $p \succ q$ and $q \succ r$ then $p \succ r$.

Note that this also requires modifying our definitions of weak preference and indifference. As shown in Dubra (2011) [9], a weak preference relation satisfying independence, and any two of the axioms of completeness, Archimedean, or mixture continuity, in the sense of Herstein and Milnor (1953) [15], must necessarily also satisfy the third.

Property (Mixture Continuity - H-M) For every $p, q, r \in \mathcal{C}$ such that $p \succ q \succ r$, the sets $\{\alpha \in [0, 1] : \alpha p + (1 - \alpha)q \succeq r\}$ and $\{\alpha \in [0, 1] : \alpha p + (1 - \alpha)q \preceq r\}$ are closed.

Therefore if completeness is relaxed then either the Archimedean axiom or mixture continuity must be violated, even though they provide similar continuity properties. To resolve this seeming contradiction, we adopt the convention of Karni (2011) [18] and use the following definition for weak preference.

Definition (Weak Preference) For every $p, q \in \mathcal{C}$, $p \succsim q$ if and only if for every $r \in \mathcal{C}$, $r \succ p$ implies $r \succ q$.

Definition (Indifference) For every $p, q \in \mathcal{C}$, $p \sim q$ if and only if $p \succsim q$ and $p \precsim q$.

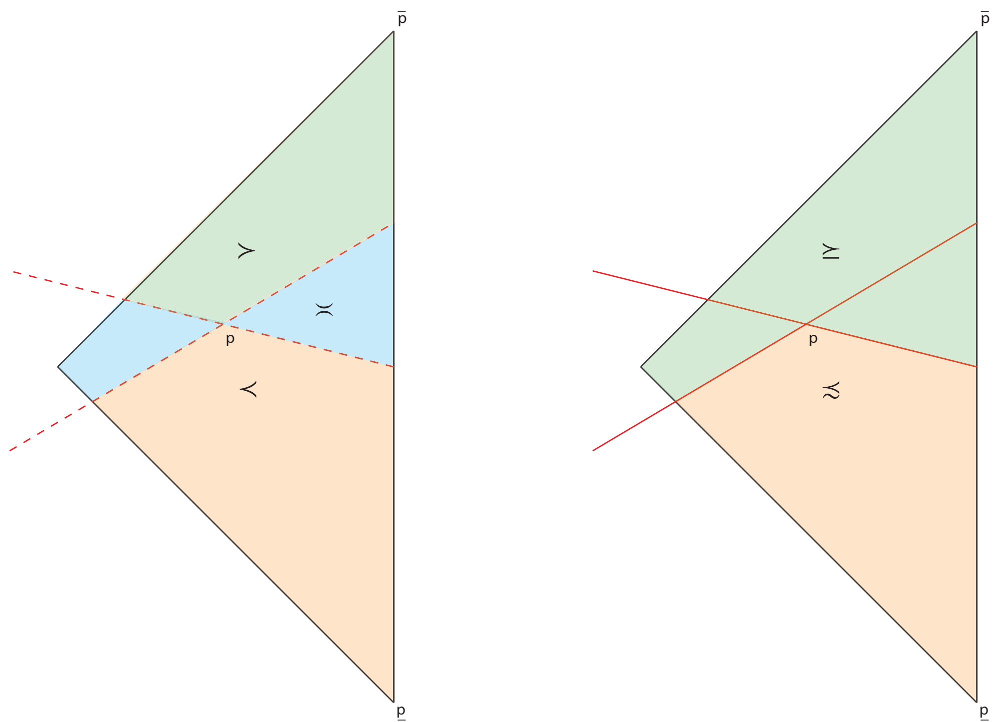
Unlike the previously defined weak preference relation $\succeq$, the new definition $\succsim$ no longer takes $\succ$ as its asymmetric part, which was the rationale for taking the strict preference as the primitive in the first place. Note that $\succsim$ is transitive by construction but complete if and only if $\succ$ is negatively transitive, while on the other hand $\succeq$ was complete by the asymmetry of $\succ$ but transitive if and only if $\succ$ is negatively transitive. Hence these two definitions agree only under completeness, otherwise $\succsim$ is a strict subset of $\succeq$ in which case the relations defined so far are insufficient to fully characterize preferences, as there may exist pairs of alternatives to which neither is even weakly preferred to the other. To accommodate this possibility, we introduce the following relation.

Definition (Incomparability) For every $p, q \in \mathcal{C}$, $p \asymp q$ if and only if $\neg(p \succ q)$ and $\neg(p \prec q)$.

The incomparability relation $\asymp$ is defined here in a weak sense, indicating only that neither lottery is strictly preferred to the other, and is an equivalence relation identical to the indifference relation $\sim$ if and only if $\succ$ is negatively transitive, otherwise it will be intransitive whenever the negation of $\succ$ is. Therefore the relation is set up to encompass all instances of indecisive behavior, including both indifference and strict incomparability. Note that this also requires a slight modification to the Archimedean axiom, as if for example $p \succ r$ and $q \asymp r$ we wish to still require that there is some mixture $\alpha p + (1 - \alpha)q \succ r$, which is not necessarily guaranteed by Axiom 2*.

Axiom 2 (Archimedean) For every $p, q, r \in \mathcal{C}$ if $p \succ r$ then there is $\alpha \in (0, 1)$ such that $\alpha p + (1-\alpha)q \succ r$ and if $q \prec r$ then there is $\beta \in (0,1)$ such that $\beta p + (1-\beta)q \prec r$.

For example, if p represents a certain payoff of \$100 and q_α represents a lottery that yields \$300 with probability α and nothing otherwise, we may have that $p \asymp q_{0.5}$ and $p \asymp q_{0.51}$, but by mixture monotonicity we must necessarily have that $q_{0.51} \succ q_{0.5}$ so that $\asymp$ is intransitive. Note that in this example since p is incomparable with two distinct mixtures, one of which is strictly preferred to the other, there cannot be any α for which $p \sim q_\alpha$ and therefore no expected utility representation of $\succ$ may exist. However, is it evidently fruitful to consider the range of α for which $p \asymp q_\alpha$ as a starting point to construct a multi-utility representation, an idea which we will now explore in greater detail.



(a) Strict Preference and Incomparability *(b)* Weak Preference

Fig. 1.7: Definitions of Preference and Incomparability

If $\mathcal{C}$ is endowed with a topology, such as $\Delta(X)$ with the Euclidean metric, then allowing for incompleteness can be understood as restricting a lottery's upper and lower contour sets, $B(p) = \{q \in \mathcal{C} : q \succ p\}$ and $W(p) = \{q \in \mathcal{C} : q \prec p\}$ respectively, only to be convex cones but not necessarily open half spaces, separated by an indifference hyperplane by which the utility function may be defined. Figure 1.7a shows that within the context of the triangle diagram, under incompleteness the incomparability set $I(p) = \{q \in \mathcal{C} : q \asymp p\}$ may now have a non-empty interior, rather than consisting only of the indifference curve intersecting p. Figure 1.7b shows how the two definitions of weak preference may now differ, with $\succsim$ representing only the closure of $\succ$ while $\succeq$ is taken as the complement of $\prec$, so that the former excludes the interior of the incomparability set while the latter includes it, with the two coinciding if and only if $\succ$ is complete and this interior is empty. Note that the actual indifference relation is typically sparse and here consists only of identical pairs, as it is given by the intersection of the boundaries of the upper and lower contour sets. This reflects our earlier assertion that the vast majority of instances where no preference is expressed reflect incomparability, carrying none of implications of equivalence that indifference represents.

If the independence axiom holds, then its substitution property can also be applied to the incomparability relation, so that $p \asymp q$ if and only if $\alpha p + (1-\alpha)r \asymp \alpha q + (1-\alpha)r$ for every α and r. Thus the incomplete preference analogue to the result of parallel indifference curves under expected utility is that the shape of the incomparability set is invariant. Intuitively, this implies that incomparability can neither be diluted nor magnified by equally shifting the proportion of the common consequences in either alternative. An individual who does not know whether she prefers coffee or tea will remain unwaveringly indecisive regardless of the probability of receiving hot chocolate. Hence, just as under completeness we can use comparisons between any lottery p and mixtures of the $\overline{p}$ and $\underline{p}$ in order to generate utility functions over the entire plane $\mathcal{L}(p)$, though now these utilities will merely be compatible with preferences, in the sense of Aumann (1962) [2], rather than representing them. For every $p \in \mathcal{C}$, define the set of incomparable mixtures as

$$A(p) = \{\alpha \in [0,1] : p \asymp \zeta_\alpha\}$$

By independence, each $\alpha \in A(p)$ defines a set of parallel incomparability curves and hence a utility $U^\alpha : \mathcal{L}(p) \mapsto \mathbb{R}$ by setting $U^\alpha(q) = \lambda\alpha + (1-\lambda)\theta$ for every $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$. If $p \succ q$, then q must lie below every incomparability curve intersecting p, so that $U^\alpha(p) > U^\alpha(q)$ for every $\alpha \in A(p)$. Therefore, collecting the utilities in $\mathcal{U}(p) = \{U^\alpha : \alpha \in A(p)\}$ defines a multi-utility representation for $\succ$ over the plane $\mathcal{L}(p)$.

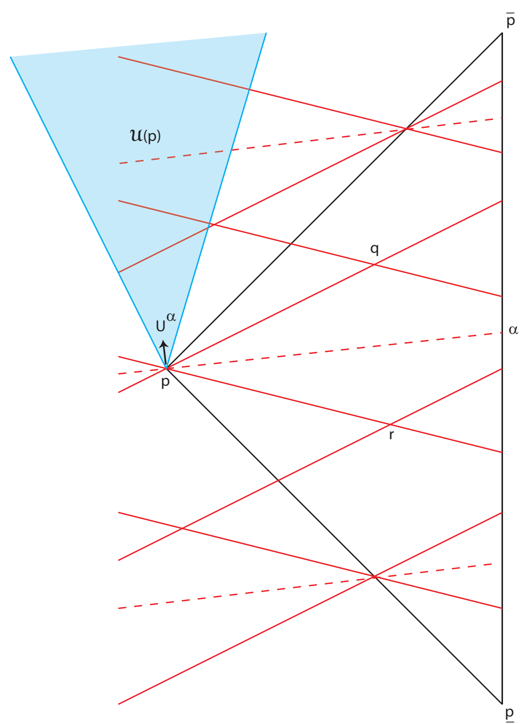
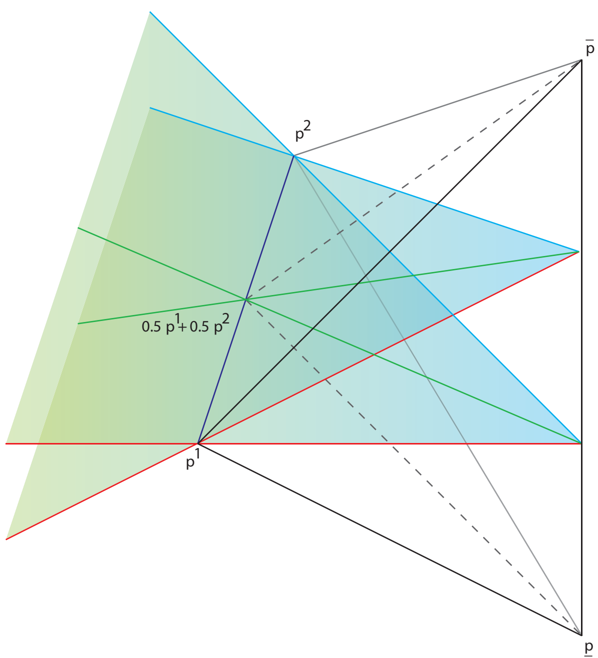


Fig. 1.8: Multiple Expected Utility

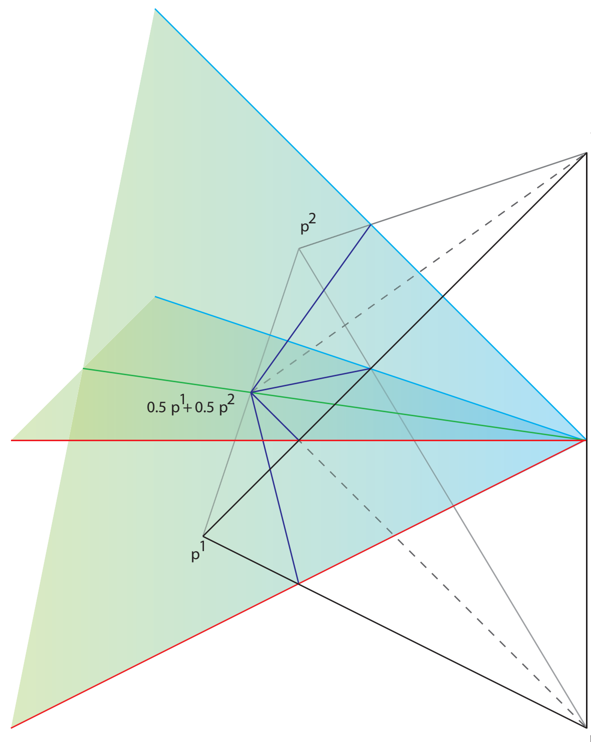
As shown in Figure 1.8, in the triangle diagram incompleteness can be interpreted as the decision maker being unsure of her attitude toward risk, so that her preferences are given by multiple sets of parallel incomparability curves of varying steepness. Demonstrating the results of Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13], the set of utilities is itself given by a closed and convex cone, as any incomparability curves of intermediate steepness between the two extremes also define valid utilities. The diagram also shows the intransitivity of the incomparability relation, as we have $p \asymp q$ and $p \asymp r$, but $q \succ r$. Intuitively, this arises because the indecision is generated by the disagreement of different decision criteria which does not obey transitivity even if each individual criterion does. Here only the most risk averse evaluation scheme that the decision maker employs considers q to be equivalent to p and every other utility judges it strictly better, while only the most risk seeking utility function assigns equal value to r and p, which allows for all of them to agree that q is strictly better than r.

Observe that at least in the two-dimensional case, the representation may be parsimoniously defined by only a pair of utilities, representing the most and least risk averse of the decision criteria corresponding to the upper and lower bounds of $A(p)$, with the entire cone given by taking linear combinations of these. As we shall see, this result does not extend to the entirety of the mixture space, as a convex cone in three or more dimensions requires arbitrarily many vectors to define and hence even a parsimoniously defined representation may comprise infinitely many utilities. Intuitively, in the two-dimensional case a utility is entirely defined by comparing p to the set of mixtures ζ_α, which conflates the decision maker's affinity for p and her risk attitude. However, if we have p^1 and p^2 that do not lie on the same $\mathcal{L}(p)$, tastes must also reflect the relative ranking of p^1 and p^2 in addition to the level of risk aversion, and may in fact judge these to be incomparable, especially if they are not taken to represent monetary values.

Recall that under completeness, the independence axiom further allowed the utility functions defined over the individual planes to be connected across each $\mathcal{L}(p)$ to yield a utility over the entirety of $\mathcal{C}$. However, the intransitivity of $\asymp$ complicates matters here since this property relied on multiple applications of independence in succession. Note that if $\alpha^i \in A(p^i)$ for $i = 1, 2$, then we must have that $\pi p^1 + (1-\pi)p^2 \asymp \pi\zeta_{\alpha^1} + (1-\pi)p^2$ and $\pi\zeta_{\alpha^1} + (1-\pi)p^2 \asymp \pi\zeta_{\alpha^1} + (1-\pi)\zeta_{\alpha^2}$, but we may have $\pi p^1 + (1-\pi)p^2 \succ \pi\zeta_{\alpha^1} + (1-\pi)\zeta_{\alpha^2}$. Therefore, since $\pi\alpha^1 + (1-\pi)\alpha^2 \notin A(\pi p^1 + (1-\pi)p^2)$, there can be no linear utility $U : \mathcal{C} \mapsto \mathbb{R}$ that coincides with U^{α^i} over $\mathcal{L}(p^i)$ for $i = 1, 2$. This implies that the utilities over $\mathcal{L}(p)$ cannot be constructed simply by arbitrarily matching the utilities defined over the individual $\mathcal{L}(p)$ with one another.



(a) "Agreement" ($p_1 \sim p_2, \frac{1}{2}p^1 + \frac{1}{2}p^2 \asymp \zeta_{\frac{1}{2}}$)



(b) "Disagreement" ($p_1 \asymp p_2, \frac{1}{2}p^1 + \frac{1}{2}p^2 \sim \zeta_{\frac{1}{2}}$)

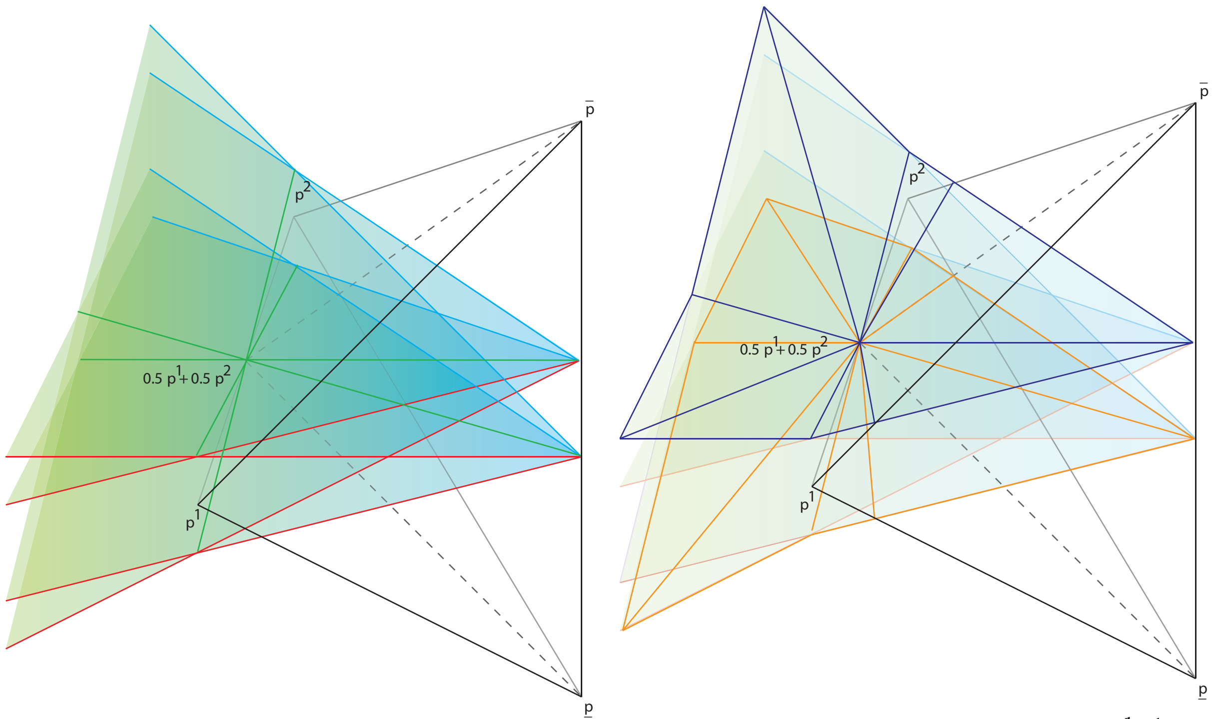
Fig. 1.9: Multiple Expected Utility in Three Dimensions

For example suppose that, as depicted in Figure 1.9, we have for $i = 1, 2$ that $A(p^i) = [\frac{1}{3}, \frac{2}{3}]$, so that each p^i takes a range of utility values, from which we can generate multi-utility representations for the restriction of $\succ$ to each $\mathcal{L}(p^i)$. Even if we assume $\succ$ has a parsimonious representation over $\mathcal{C}$ consisting of only a pair of utilities $\mathcal{U} = \{U^1, U^2\}$, we are unable to determine it by simply observing the $A(p^i)$ alone. Figure 1.9a shows the case where $U^1(p^1) = U^1(p^2) = \frac{2}{3}$ and $U^2(p^1) = U^2(p^2) = \frac{1}{3}$, so that both of the decision criteria agree that p^1 and p^2 are equally good and hence $p^1 \sim p^2$. Thus for every $\pi \in [0,1]$, we have $U^1(\pi p^1 + (1-\pi)p^2) = \frac{2}{3}$ and $U^2(\pi p^1 + (1-\pi)p^2) = \frac{1}{3}$, and hence $A(\pi p^1 + (1-\pi)p^2) = [\frac{1}{3}, \frac{2}{3}]$ and $p^1 \sim \pi p^1 + (1-\pi)p^2 \sim p^2$. Intuitively, in this case the utilities simply represent different levels of risk aversion, as the decision maker considers p^1 and p^2 to be identical but is unable to compare either of them against a range of mixtures ζ_α. On the other hand, Figure 1.9b shows the case in which $U^1(p^1) = U^2(p^2) = \frac{2}{3}$ and $U^1(p^2) = U^2(p^1) = \frac{1}{3}$, so that the utilities have conflicting valuations of these elements, with U^1 judging p^1 to be strictly better and U^2 indicating the opposite, and $p^1 \asymp p^2$ arising from this disagreement. In this case, we have that $U^1(\pi p^1 + (1-\pi)p^2) = \frac{2}{3} - \frac{1}{3}\pi$ and $U^2(\pi p^1 + (1-\pi)p^2) = \frac{1}{3} + \frac{1}{3}\pi$, so that we have that $A(\frac{1}{2}p^1 + \frac{1}{2}p^2) = \{\frac{1}{2}\}$ and hence $\frac{1}{2}p^1 + \frac{1}{2}p^2 \sim \zeta_{\frac{1}{2}}$. Here it is not risk attitude that varies between U^1 and U^2 but the relative valuations of p^1 and p^2, with the decision criteria just so happening to exactly agree on how to evaluate a mixture of these in equal proportion.

Note that in both of these cases, the upper and lower contour sets are wedge-shaped, though oriented differently in either instance, as the cones are formed by a pair of incomparability planes thus allowing for a non-trivial indifference relation to be defined by their intersection. However, there exist an entire range of possible representations between these extreme cases of total agreement and symmetric disagreement, as neither took into consideration elements in the interior of either $A(p^i)$. For example, suppose for example that the parsimonious representation of $\succ$ requires four utility functions which take the values given below.

	U^1	U^2	U^3	U^4
p^1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
p^2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{2}p^1 + \frac{1}{2}p^2$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{5}{12}$	$\frac{5}{12}$

Here we have that $A(p^1) = A(p^2) = [\frac{1}{3}, \frac{2}{3}]$ as before, but $A(\frac{1}{2}p^1 + \frac{1}{2}p^2) = [\frac{5}{12}, \frac{7}{12}]$. As



(a) Incomparability Planes at $\frac{1}{2}p^1 + \frac{1}{2}p^2$

(b) Upper and Lower Contour Sets of $\frac{1}{2}p^1 + \frac{1}{2}p^2$

Fig. 1.10: Multiple Expected Utility in Three Dimensions with Four Utilities

shown in Figure 1.10, the upper and lower contour sets intersect only at $\frac{1}{2}p^1+\frac{1}{2}p^2$ itself, so that there are no other lotteries to which it is indifferent. Figure 1.10a shows that the set of utilities $\mathcal{U} = \{U^1, U^2, U^3, U^4\}$ is generated by matching utilities over $\mathcal{L}(p^1)$ and $\mathcal{L}(p^2)$ corresponding to three levels of risk aversion, including the intermediate value represented by $\alpha = \frac{1}{2}$ that was not relevant in either of the previous examples. Figure 1.10b shows that the incomparability plane intersecting p corresponding to any utility in $\mathcal{U}$ is a supporting hyperplane for both $B(p)$ and $W(p)$, and furthermore that plane corresponding to a utility in the closed convex cone generated by the elements of $\mathcal{U}$ is a separating hyperplane between $B(p)$ and $W(p)$.

Given that under the independence axiom, the upper and lower contour sets retain their shapes throughout $\mathcal{C}$, reversing this principle gives us a procedure for finding the representation. Following Baucells and Shapley (1998) [3], Dubra, Maccheroni, and Ok (2004) [10], and Galaabaatar and Karni (2012) [13], construct the domination cone $\mathcal{D} = \{\lambda(p-q) : p \succ q, \lambda \geq 0\}$, so that the set of utilities $\mathcal{U}$ is given by its dual cone. Note that this result is entirely dependent on the independence axiom, which ensures that p is strictly preferred to q if and only if $p - q \in \mathcal{D}$ and consequently that

$$p \succ q \Leftrightarrow U(p) > U(q) \quad \forall\, U \in \mathcal{U}$$

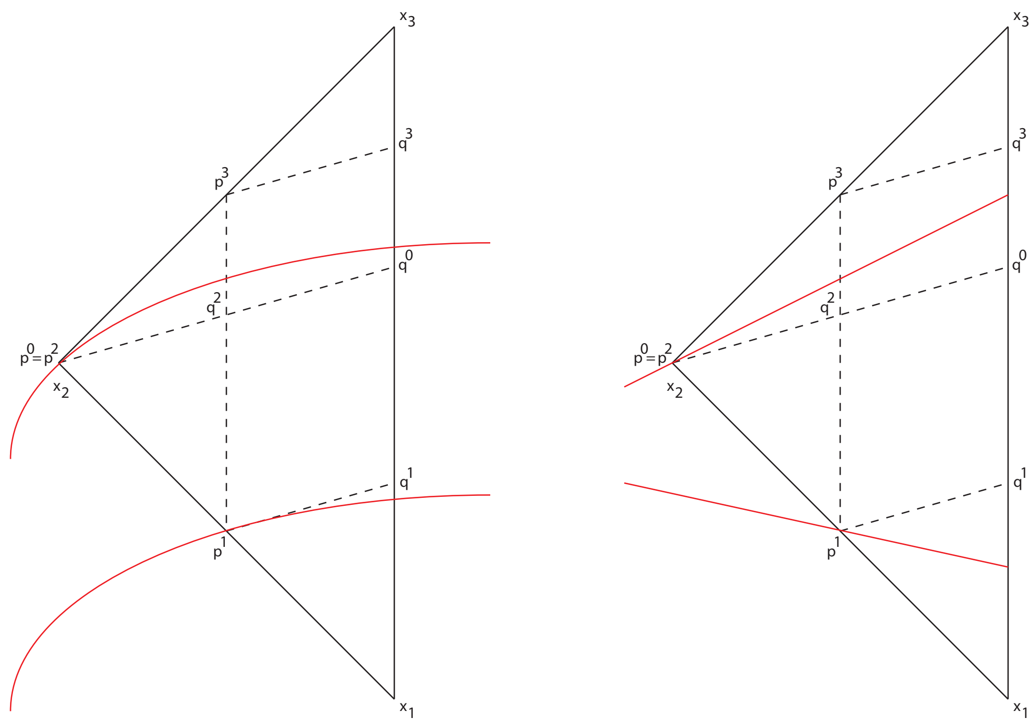
If $\mathcal{C} = \Delta(X)$, define for every $U \in \mathcal{U}$ the utility over outcomes $u : X \mapsto \mathbb{R}$ such that $u(x) = U(\delta_x)$. Collecting these in $\tilde{\mathcal{U}} = \{u : U \in \mathcal{U}\}$, we have that

$$p \succ q \Leftrightarrow \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)u(x) \quad \forall\, u \in \tilde{\mathcal{U}}$$

1.2.3 Non-Expected Utility

We now switch our attention to relaxing the independence axiom. Recall that the independence axiom required that the triangle diagram consist of a single set of parallel indifference curves under completeness, and multiple sets of parallel incomparability curves under incompleteness. Therefore, violations of independence as in the Allais paradox must imply that these curves have a different structure. Following the construction of Chew (1989) [6], let $\mathcal{C} = \Delta(X)$ for $|X| = 3$, and define $p^0 = \delta_{x_2}$ and $q^0 = \alpha\delta_{x_3} + (1-\alpha)\delta_{x_1}$ for some $\alpha \in [0,1]$. For some $\beta \in [0,1]$ and $i = 1,2,3$, let $p^i = \beta p^0 + (1-\beta)\delta_{x_i}$ and $q^i = \beta q^0 + (1-\beta)\delta_{x_i}$. Therefore, by assigning monetary payoffs to the outcomes of $\{x_1, x_2, x_3\}$=\{\$0,\$1m,\$5m\}, we can construct the Allais paradox for some choice of α and β. The common consequence effect is given by $(\alpha, \beta) = (\frac{10}{11}, 0.11)$ and preferences of $p^2 \succ q^2$ and $p^1 \prec q^1$, and the common ratio effect by $(\alpha, \beta) = (0.8, 0.05)$ and preferences of $p^0 \succ q^0$ and $p^1 \prec q^1$.

As shown in Figure 1.11, as each of the vectors $p^i - q^i$ are parallel, preferences that



(a) Rank-Dependent Utility *(b)* Non-Expected Utility

Fig. 1.11: Allais Paradox

adhere to the Allais paradox do not have an expected utility representation, and thus the indifference curves must either be non-linear or non-parallel. As shown in Chew, Karni, and Safra (1987) [5], in the context of rank-dependent utility, preferences exhibit risk aversion if both the utility and probability transformation functions are concave, so that as demonstrated in Figure 1.11a the indifference curves will be concave themselves, displaying an overall aversion to mixture of any kind. On the other hand, if preferences adhere to the betweenness axiom and exhibit increasing risk aversion in first-order stochastically dominant shifts as in Machina (1982) [22], then as shown in Figure 1.11b, the indifference curves will be linear but fan out moving upwards in the direction of $\delta_{x_3} - \delta_{x_1}$. Either of these may explain Allais-type behavior, but for our purposes we will restrict our attention to the class of betweenness-conforming preferences, as adapting rank-dependent models to account for incompleteness will prove troublesome if the decision maker is unable to actually rank the outcomes.

In lieu of independence, make the following assumptions on $\succ$.

Axiom 3 (Mixture Dominance) For every $p, q, r \in \mathcal{C}$ and $\alpha \in (0,1)$, if $p \succ r$ and $q \succ r$ then $\alpha p + (1-\alpha)q \succ r$, and if $p \prec r$ and $q \prec r$ then $\alpha p + (1-\alpha)q \prec r$.

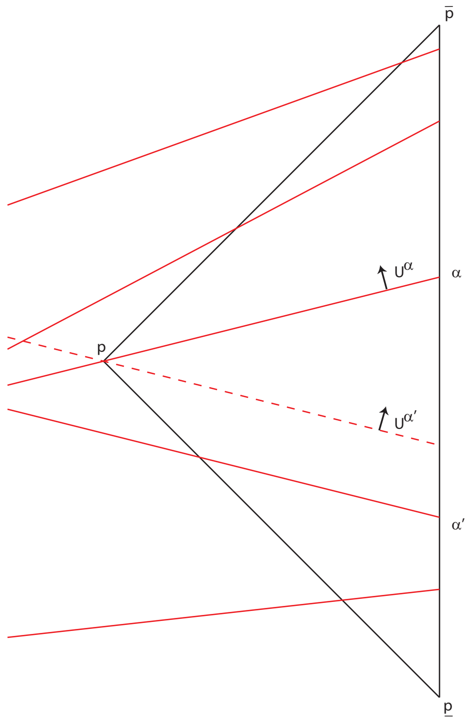
Axiom 4 (Betweenness) For every $p, q \in \mathcal{C}$ and $\alpha \in (0,1)$, $p \succ q$ if and only if $p \succ \alpha p + (1-\alpha)q \succ q$.

Note that Axioms 3 and 4 imply that the upper and lower contour sets of any mixture are convex cones, without restricting them to have the same shape everywhere. Together with Axiom 2 they maintain the useful properties of mixture continuity and mixture monotonicity which were provided by independence in expected utility.

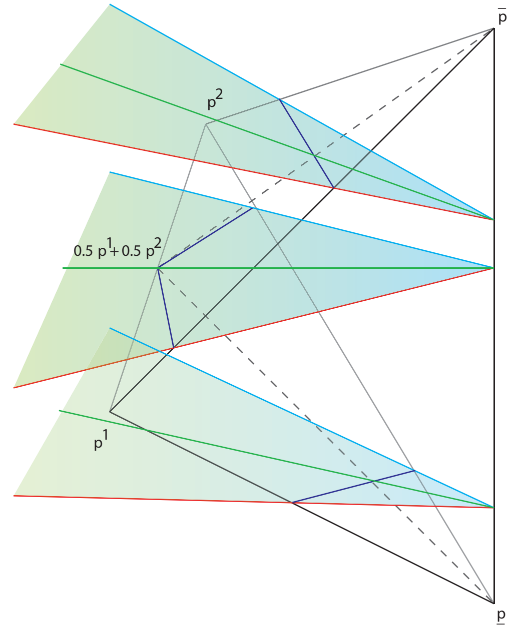
Lemma 1 (Mixture Continuity) If $\succ$ satisfies Axioms 2 and 3, then for every $p, q, r \in \mathcal{C}$, the sets $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$ and $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \prec r\}$ are open, and $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \asymp r\}$ is closed.

Lemma 2 (Mixture Monotonicity) If $\succ$ satisfies Axiom 4, then for every $p, q \in \mathcal{C}$ and $\alpha, \beta \in (0,1)$ such that $\alpha > \beta$, if $p \succ q$ then $\alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q$, and if $p \asymp q$ then $\alpha p + (1-\alpha)q \asymp \beta p + (1-\beta)q$.

Therefore under betweenness, preferences must be monotonic in any direction that exhibits either strict preference, indifference, or incomparability, so that in a triangle diagram the indifference curves must be straight lines with increasing preference in the direction of $\overline{p} - \underline{p}$. However, since we no longer have the strong substitution property of the independence axiom, these lines no longer need to be parallel. Fishburn (1983) [12] showed that complete preferences satisfying betweenness have a representation by a continuous utility function $U : \mathcal{C} \mapsto \mathbb{R}$ where $U(\pi p + (1-\pi)q)$ is increasing, but not necessarily linear, in π if $p \succ q$ and constant if $p \sim q$. Dekel (1986) [8] further showed that this utility is characterized by a function $V : \mathcal{C} \times [0,1] \mapsto \mathbb{R}$ that is linear in its first argument and continuous in its second, so the utility is given by finding for each p the α that solves $V(p, \alpha) = \alpha$ and setting $U(p) = \alpha$.



(a) Two Dimensions

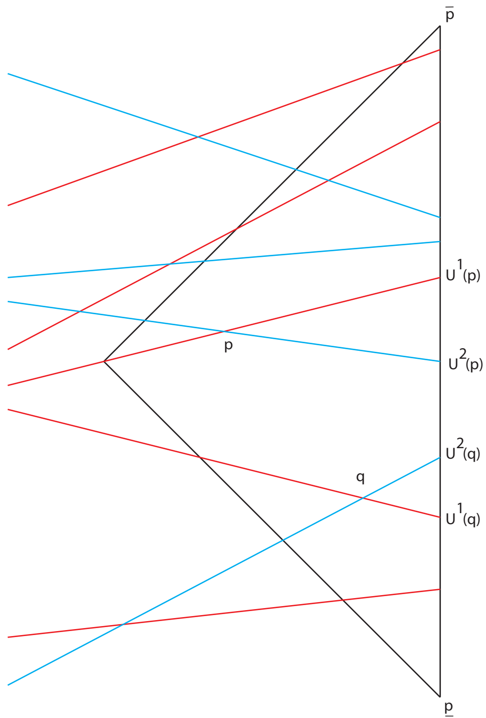


(b) Three Dimensions

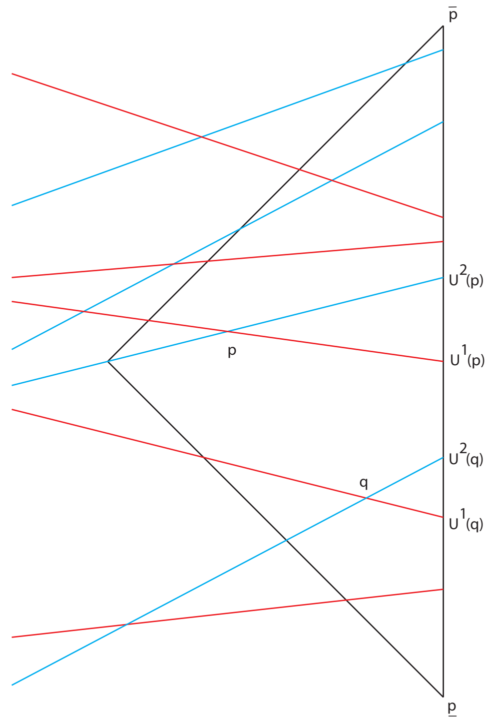
Fig. 1.12: Non-Expected Utility

Figure 1.12 shows the triangle diagram under non-expected utility, where the representation can be thought of as defining functions $U^\alpha : \mathcal{L}(p) \mapsto \mathbb{R}$ for $\alpha \in [0, 1]$ such that each U^α generates a set of parallel lines of the same steepness as the indifference curve intersecting ζ_α, and setting $U(q) = U^\alpha(q)$ for every mixture lying along this line. This interpretation bears some similarities to that of multi-utility representations, with the set of U^α denoting decision criteria representing different tastes, except that the decision maker evaluates each lottery by choosing one of these utilities, rather than evaluating every lottery by all of them. In general, preferences are given by a set of indifference hyperplanes which are not necessarily parallel, with each mixture lying on such a plane assigned the utility value by the point ζ_α at which it intersects the line connecting the best and worst elements.

However, attempting to adapt the model of betweenness-conforming preferences to



(a) Possible "Grouping" of Incomparability Curves



(b) Alternate "Grouping" of Incomparability Curves

Fig. 1.13: Non-Expected Utility without Completeness

admit incompleteness may prove troublesome, since multi-utility models depend on the substitution property of the independence axiom to provide structure from which the representation can be constructed from a sparse set of comparisons, which is lacking here. Figure 1.13 depicts the triangle diagram under preferences that satisfy neither completeness nor independence, consisting of multiple sets of incomparability curves which are not necessarily parallel. Note however that without any sort of assumption that relates these curves to one another, actually defining the utilities themselves proves difficult, since we cannot simply use the range of utility values for p to generate the preferences over the entirety of $\mathcal{L}(p)$ as before. The matching problem now manifests in two dimensions as well, as Figures 1.13a and 1.13b show two different and equally valid groupings of incomparability curves into utility functions. As the shapes of the upper and lower contour sets may vary arbitrarily, attempting to define utilities in terms of their supporting hyperplanes will prove fruitless as well.

1.2.4 Weighted Expected Utility

Evidently, in order to model preferences that satisfy neither independence nor completeness, we need to consider making another assumption that provides some sort of substitution property, yet does not restrict the incomparability curves to be parallel. In any event, we would gain little insight if the degree of incompleteness were allowed to vary arbitrarily. Recall that when we observe fanning out property of Machina (1982) [22], the indifference curves appear to converge at a single point lying outside the triangle, so we now consider the assumption of Chew and MacCrimmon (1979) [7] which gives exactly this property.

Axiom 5 (Weak Substitution) For every $p, q \in \mathcal{C}$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that for every $r \in \mathcal{C}$, $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$.

Note that Axiom 5 is slightly modified to use the incomparability relation rather than indifference, as our objective is to extend the framework provided by weighted expected utility theory to admit incompleteness, but as $\asymp$ is transitive whenever $\succ$ is negatively transitive this axiom is equivalent to their assumption when $\succ$ is complete. The weak substitution principle requires any pair of alternatives for which an individual exhibits no decisive preference to maintain this relation if both are mixed with any third lottery in some fixed proportions given by β and γ, without requiring that these proportions be identical as under independence where $\beta = \gamma$. Note that unlike the independence axiom there is no equivalent version of this property that applies to either the strict or weak preference. Intuitively, it implies that even though the decision maker may consider p and q to be incomparable, she may consider them to have disparate degrees of influence on her perception of mixtures in which they are components. Hence, for some β and r she may perceive $\beta p + (1-\beta)r$ to be more similar to p than $\beta q + (1-\beta)r$ is to q, requiring that the latter be moved closer to q to compensate, so that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$ for some $\gamma > \beta$. The following lemma, from Chew (1989) [6], shows that this property may be equivalently expressed by assigning for each pair of incomparable lotteries a ratio $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$ giving the weight that the decision maker attaches to p relative to q.

Lemma 3 (Ratio Substitution) If $\succ$ satisfies Axioms 1-5, then for every $p, q \in \mathcal{C}$, if $p \asymp q$ then there exists $\tau \geq 0$ such that for every $\beta \in (0,1)$ and $r \in \mathcal{C}$, $\beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$.

Note that this odds ratio τ is necessarily unique under completeness, since if $\tau' < \tau$ then we have that $\frac{\beta\tau' q + (1-\beta)\overline{p}}{\beta\tau' + (1-\beta)} \succ \frac{\beta\tau q + (1-\beta)\overline{p}}{\beta\tau + (1-\beta)} \sim \beta p + (1-\beta)\overline{p}$. This is not necessarily the case if $\succ$ is incomplete however, so define for every p, q,

$$T(p,q) = \left\{\tau \geq 0 : \beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)} \; \forall\, \beta \in (0,1), r \in \mathcal{C}\right\}$$

If $p \asymp q$, then any $\tau \geq 0$ satisfying ratio substitution between them uniquely defines a source point o from which a set of incomparability curves originates, since for every $\beta \in (0,1)$ and $r \in \mathcal{C}$ we have by betweenness that for every $\lambda \in \mathbb{R}$,

$$\begin{aligned} \beta p + (1-\beta)r &\asymp \lambda[\beta p + (1-\beta)r] + (1-\lambda)\left[\frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}\right] \\ &= \beta\left[\lambda p + \frac{(1-\lambda)\tau q}{\beta\tau + (1-\beta)}\right] + (1-\beta)\left[\lambda + \frac{(1-\lambda)}{\beta\tau + (1-\beta)}\right] r \end{aligned}$$

The source point is defined as the intersection of all of the incomparability curves connecting $\beta p + (1-\beta)r$ to $\frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$ for each β and r, and can thus be located by finding the value λ^* that sets the second term of the right hand side in the expression above to zero.

$$\lambda^* = -\frac{1-\lambda^*}{\beta\tau + (1-\beta)} = \frac{1}{\beta(1-\tau)}$$

This implies that

$$o = \beta \left[\lambda^* p + \frac{(1-\lambda^*)\tau q}{\beta\tau + (1-\beta)} \right] = \frac{1}{1-\tau} p - \frac{\tau}{1-\tau} q = \frac{1}{1-\frac{1}{\tau}} q - \frac{\frac{1}{\tau}}{1-\frac{1}{\tau}} p$$

Therefore, the source point o lies on the extension of the line connecting the incomparable alternatives p and q, and is placed to the left of p if $\tau < 1$ and to the right of q if $\tau > 1$, approaching a point infinitely distant as τ approaches unity from either direction. As $\tau = 1$ indicates that p and q satisfy the strong substitution property of independence, the incomparability curves that o projects will be parallel in this case, thus obeying the results of expected utility. Equivalently, since τ satisfies ratio substitution if and only if $\gamma = \frac{\beta\tau}{\beta\tau + (1-\beta)}$ satisfies weak substitution for every β, we have that

$$o = \frac{\beta(1-\gamma)}{\beta - \gamma} p - \frac{\gamma(1-\beta)}{\beta - \gamma} q$$

Hence were the source point itself considered a lottery, it would be deemed incomparable to any alternative, which can be demonstrated simply by drawing an incomparability curve originating from o through it.

Figure 1.14 shows the nature of preferences if independence is relaxed to weak substitution. Here the decision maker is indifferent between p and some mixture of the best and worst outcomes $q = \zeta_\alpha$, but has a distorted perception of mixture that under-weights p relative to q. Hence she perceives both $\beta p + (1-\beta)\overline{p}$ to be closer to $\overline{p}$, and hence better, than $\beta q + (1-\beta)\overline{p}$ is, while viewing $\beta p + (1-\beta)\underline{p}$ as closer to $\underline{p}$ and worse than $\beta q + (1-\beta)\underline{p}$. Thus some $\tau < 1$ satisfies ratio substitution, or equivalently that weak substitution is satisfied for some $\gamma < \beta$. This implies that the indifference curves converge at a source $o = \frac{p - \tau q}{1-\tau}$ lying to the left of p, and hence obey the fanning out property, becoming steeper and exhibiting increasing risk aversion as

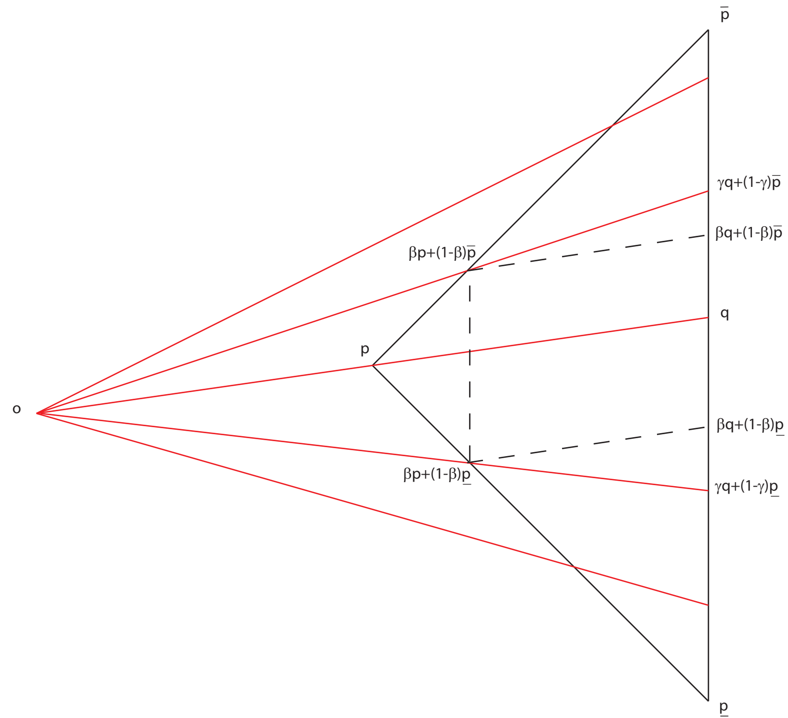


Fig. 1.14: Weighted Expected Utility

prospects improve in terms of an upward shift in the direction $\overline{p} - \underline{p}$. Alternatively, if some $\tau > 1$ satisfied ratio substitution, then p would be over-weighted relative to q and the source o would be located to the right of q, so that the indifference curves would instead fan in and the decision maker's risk attitude would vary in the opposite manner.

While the Allais paradox implies that indifference curves should fan out in the lower half of the simplex, given that it relates to mixtures of p and q with the worst element $\underline{p}$, it is not quite so clear that this same pattern should continue in the upper half as well. Note that the interpretation of the effect in the lower half relies on the decision maker viewing p as the "safe" alternative, so that she becomes more willing to exchange it for the risky bet q as the the probability of receiving the worst possible outcome $\underline{p}$ increases. In the upper half however, if the probability of receiving the best element $\overline{p}$ is sufficiently high, then the choice may instead be framed as a possibility of loss from a default of $\overline{p}$, in which case receiving p is no longer seen as a gain but a disappointment that the decision maker wishes to avoid. Gul (1991) [14] proposes a theory of disappointment aversion under which indifference curves become shallower moving in either direction, fanning out on the lower half but fanning in on the upper half.

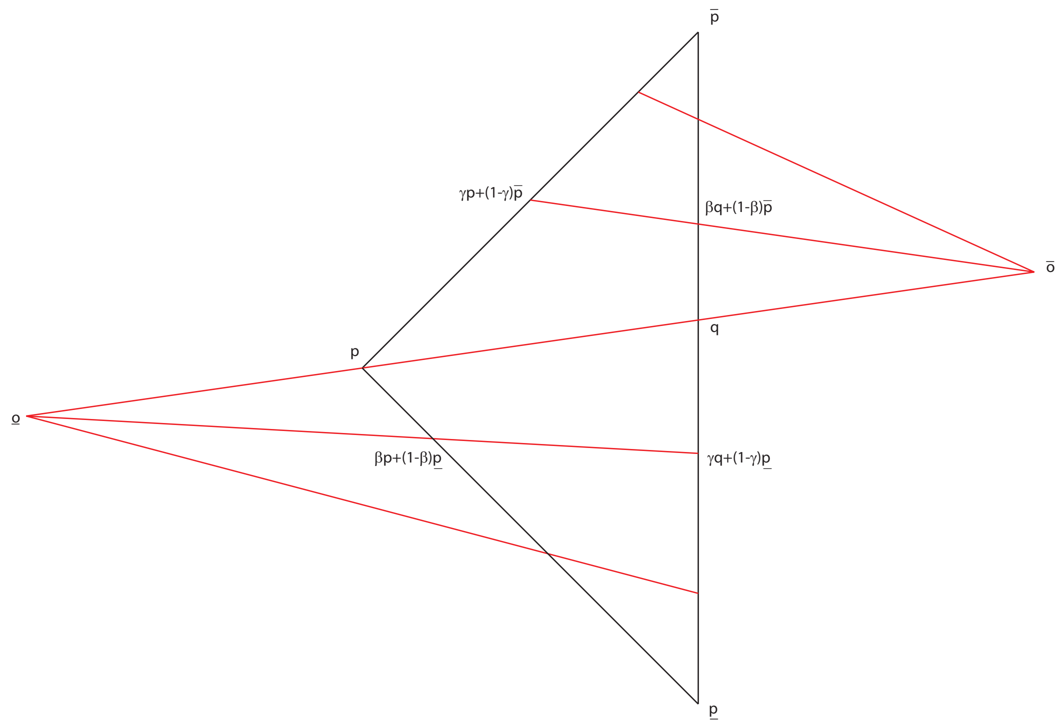


Fig. 1.15: Disappointment Aversion

As shown in Figure 1.15, preferences under disappointment aversion are given by indifference curves projected from a pair of source points. If $p \sim q = \zeta_\alpha$, then under the symmetry assumption there is $\tau \geq 0$ such that for every β we have that $\beta p + (1-\beta)r \sim \frac{\beta\tau q+(1-\beta)r}{\beta\tau+(1-\beta)r}$ for $r \prec p$ and $\beta p + (1-\beta)r \sim \frac{\beta\frac{1}{\tau}q+(1-\beta)r}{\beta\frac{1}{\tau}+(1-\beta)r}$ for $r \succ p$, with $\rho = \frac{1-\tau}{\tau}$ giving the disappointment aversion parameter. Hence if $\tau < 1$ and thus $\rho > 0$ as expected, then the indifference curves in the lower half are projected from $\underline{o} = \frac{1}{1-\tau}p - \frac{\tau}{1-\tau}q$ lying to the left of p and fan out, while those in the upper half are projected from $\overline{o} = \frac{1}{1-\frac{1}{\tau}}p - \frac{\frac{1}{\tau}}{1-\frac{1}{\tau}}q$ lying to the right of q and fan in. Therefore, the degree of risk aversion decreases moving away from p in either direction.

Chew (1989) [6] provides an alternative formulation that does not impose this sym-

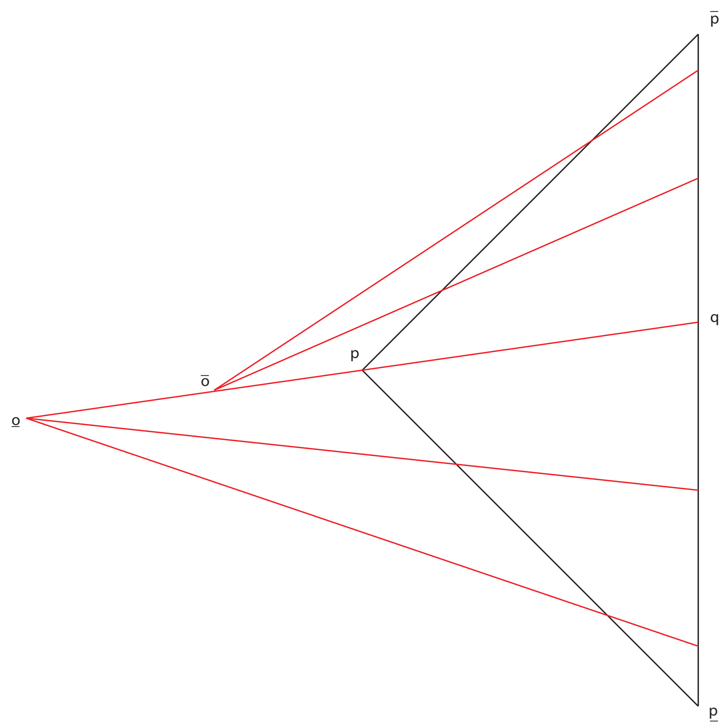


Fig. 1.16: Semi-Weighted Expected Utility

metry condition, showing that if ratio substitution is weakened to allow the odds ratio $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$ to depend on the third alternative r being mixed in, then the indifference curves are exactly those as in Dekel (1986) [8]. If the dependence of τ on r is allowed only for degenerate lotteries then, as depicted in Figure 1.16, there exist a pair of source points $\overline{o}$ and $\underline{o}$ arbitrarily located on the extended line between p and q, rather than reflections of each other as under disappointment aversion.

This class of semi-weighted utility models thus induces a kink in the state-contingent indifference map along the certainty line, as depicted in Figure 1.17. For our purposes however, we will maintain the basic form of the ratio substitution property where the odds ratio τ is independent of r. Under incompleteness we cannot simply partition the simplex into upper and lower halves since depending on how these are defined they would either overlap at or omit the set of incomparable lotteries, and given the effects of relaxing completeness in the expected utility model, it is rather self-evident that the notion of multiple source points should naturally arise anyway.

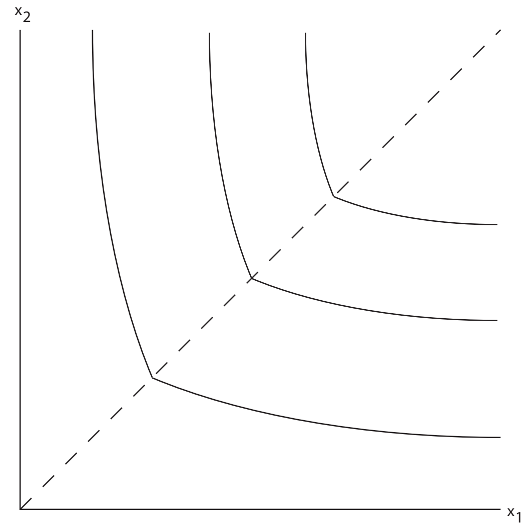


Fig. 1.17: State-Contingent Diagram of Semi-Weighted Expected Utility

To derive the weighted expected utility representation note that, as under expected utility, the structure of indifference map for any plane $\mathcal{L}(p)$ is generated simply by investigating p itself, though here both its utility and weight need to be set to define these values over the entirety of $\mathcal{L}(p)$. Under completeness, the indifference curves originating from the source point partition $\mathcal{L}(p)$ into equivalence classes so that defining $U : \mathcal{L}(p) \mapsto \mathbb{R}$ such that $p \sim \zeta_{U(p)}$, we have that $p \succ q$ if and only if $U(p) > U(q)$, though U is not necessarily a linear function as under expected utility. Define the weight function $W : \mathcal{L}(p) \mapsto \mathbb{R}$ such that for every $q \in \mathcal{L}(p)$, $W(q)$ is the odds ratio $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$ satisfying substitution between q and $\zeta_{U(q)}$, so that for every β and r we have

$$\beta q + (1-\beta) r \sim \frac{\beta W(q) \zeta_{U(q)} + (1-\beta) r}{\beta W(q) + (1-\beta)}$$

We can easily check that U is a weighted linear function, taking W as its linear weight function. For every p, q and $\pi \in [0,1]$, since $p \sim \zeta_{U(p)}$, we have by ratio substitution that for every β and r that

$$\beta[\pi p + (1-\pi)q] + (1-\beta)r \sim \frac{\beta\pi W(p)\zeta_{U(p)} + \beta(1-\pi)q + (1-\beta)r}{\beta\pi W(p) + \beta(1-\pi) + (1-\beta)}$$

Furthermore, since $q \sim \zeta_{U(q)}$ we have by another application of ratio substitution that

$$\frac{\beta\pi W(p)\zeta_{U(p)} + \beta(1-\pi)q + (1-\beta)r}{\beta\pi W(p) + \beta(1-\pi) + (1-\beta)} \sim \frac{\beta\pi W(p)\zeta_{U(p)} + \beta(1-\pi)W(q)\zeta_{U(q)} + (1-\beta)r}{\beta\pi W(p) + \beta(1-\pi)W(q) + (1-\beta)}$$

Hence by the transitivity of the indifference relation,

$$\begin{aligned}\beta(\pi p + (1-\pi)q) + (1-\beta)r &\sim \frac{\beta\pi W(p)\zeta_{U(p)} + \beta(1-\pi)W(q)\zeta_{U(q)} + (1-\beta)r}{\beta\pi W(p) + \beta(1-\pi)W(q) + (1-\beta)} \\ &= \frac{\beta[\pi W(p) + (1-\pi)W(q)]\zeta_{\frac{\pi W(p)U(p)+(1-\pi)W(q)U(q)}{\pi W(p)+(1-\pi)W(q)}} + (1-\beta)r}{\beta[\pi W(p) + (1-\pi)W(q)] + (1-\beta)}\end{aligned}$$

Thus we have that the utility and weight functions (U, W) form a weighted linear pair

$$\begin{aligned}U(\pi p + (1-\pi)q) &= \frac{\pi W(p)U(p) + (1-\pi)W(q)U(q)}{\pi W(p) + (1-\pi)W(q)} \\ W(\pi p + (1-\pi)q) &= \pi W(p) + (1-\pi)W(q)\end{aligned}$$

Figure 1.18 shows the interpretation of the utility and weight functions in the two-dimensional diagram. The pair (U, W) is defined such that $o = \frac{q - W(q)\zeta_{U(q)}}{1-W(q)}$ for every $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$. Hence, the utility value is found by extending a line from o through q to intersect the best-worst line at $\zeta_{U(q)}$, and the weight taken as the ratio of the distance between o and q and distance between o and $\zeta_{U(q)}$. Thus, lotteries lying on any vertical line parallel to $\overline{p} - \underline{p}$ have equal weight, while any mixture ζ_α on the best-worst line itself is assigned unit weight under this scheme, so that in addition to normalizing the utilities such that $U(\overline{p}) = 1$ and $U(\underline{p}) = 0$ as in expected utility, this

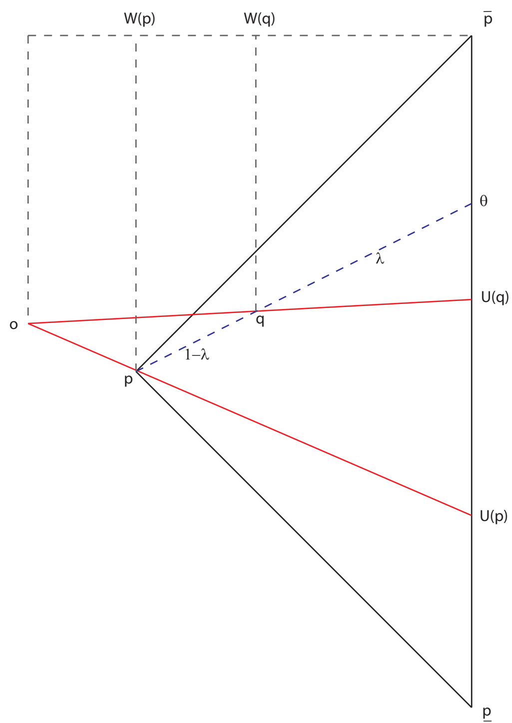


Fig. 1.18: Interpretation of Weighted Utility

formulation also normalizes the weights such that $W(\overline{p}) = W(\underline{p}) = 1$.

Now consider extending this representation to the entirety of $\mathcal{C}$. Suppose we have $p^1, p^2 \in \mathcal{C}$ such that $p^1 \sim p^2 \sim \zeta_\alpha$, then for $i = 1, 2$ we have that there is some $\tau^i \geq 0$ satisfying ratio substitution between p^i and ζ_α, and consequently a source point located at $o^i = \frac{p^i - \tau^i \zeta_\alpha}{1 - \tau^i}$. Furthermore, we have by two applications of ratio substitution that for any β and r that

$$\beta p^1 + (1-\beta) r \sim \frac{\beta \tau^1 \zeta_\alpha + (1-\beta) r}{\beta \tau^1 + (1-\beta)} \sim \frac{\beta \frac{\tau^1}{\tau^2} p^2 + (1-\beta) r}{\beta \frac{\tau^1}{\tau^2} + (1-\beta)}$$

Therefore we have that $\tau^* = \frac{\tau^1}{\tau^2}$ in turn satisfies ratio substitution between p^1 and p^2, thus clarifying the notion of the odds ratio representing the relative weight between two lotteries, as $\tau^i = W(p^i)$ for $i = 1, 2$ and hence $\tau^* = \frac{W(p^1)}{W(p^2)}$. Moreover, this implies that there is a third source point located at

$$
\begin{aligned}
o^* = \frac{p^1 - \tau^* p^2}{1 - \tau^*} &= \frac{\tau^2 p^1 - \tau^1 p^2 - \tau^1\tau^2\zeta_\alpha + \tau^1\tau^2\zeta_\alpha}{\tau^2 - \tau^1} = \frac{\tau^2(p^1 - \tau^1\zeta_\alpha) - \tau^1(p^2 - \tau^1\zeta_\alpha)}{\tau^2 - \tau^1} \\
&= \frac{\tau^2(1-\tau^1)}{\tau^2 - \tau^1}\frac{p^1 - \tau^1\zeta_\alpha}{1-\tau^1} - \frac{\tau^1(1-\tau^2)}{\tau^2-\tau^1}\frac{p^2 - \tau^2\zeta_\alpha}{1-\tau^2} = \frac{(1-\tau^1)o^1 - \tau^*(1-\tau^2)o^2}{1-\tau^*}
\end{aligned}
$$

Hence, given the three mixtures p^1, p^2, and ζ_α that define an indifference plane, the three source points that define their pairwise substitutability must all lie on the same line. This source line projects a set of indifference planes over the three-dimensional space comprising all of the planes $\mathcal{L}(\pi p^1 + (1-\pi)p^2)$ for $\pi \in [0,1]$.

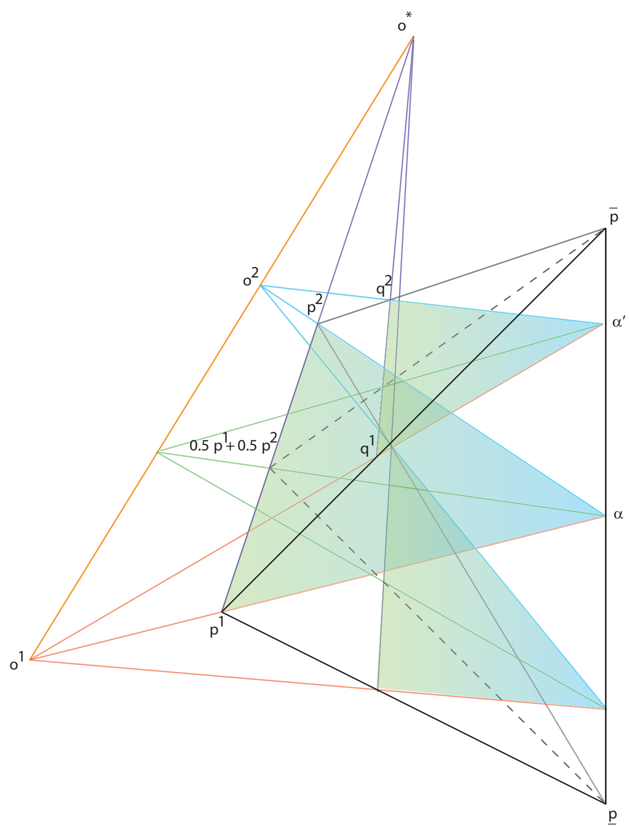


Fig. 1.19: Weighted Expected Utility in Three Dimensions

Figure 1.19 depicts weighted utility in three dimensions, so that the indifference plane given by p^1, p^2, and ζ_α defines three source points o^1, o^2, and o^* lying on the same line. Note that o^* lies to the right of p^2 implying that $\tau^* > 1$ and hence $\tau^1 > \tau^2$. For any $\alpha' \in [0,1]$, define $\beta = \frac{1-\alpha'}{1-\alpha}$ so that $\zeta_{\alpha'} = \beta\zeta_\alpha + (1-\beta)\overline{p}$. Then letting

$q^i = \frac{\beta \frac{1}{\tau^i} p^i + (1-\beta)\overline{p}}{\beta \frac{1}{\tau^i} + (1-\beta)}$ for $i = 1, 2$, we have that q^1, q^2, and ζ'_α define another indifference plane projected from the same source line. Conversely, the source line itself is found by taking the intersection of any two indifference planes, so that the planes are parallel if and only if the odds ratio between any pair of lotteries lying on either is fixed at unity, as under the independence axiom. In general, a mixture space of dimension n consists of indifference hyperplanes of dimension $n-1$, projected from a source space of dimension $n-2$, so that any k-tuple of pairwise indifferent lotteries is projected from a set of $\frac{k(k-1)}{2}$ source points that lie on a hyperplane of dimension $k-2$.

Thus if we have a basis $\{p^1, \ldots, p^n\}$ of $\mathcal{C}$ and $(U(p^i), W(p^i)) = (\alpha^i, \tau^i)$ such that source points are located at $o^i = \frac{p^i - \tau^i \zeta_{\alpha^i}}{1-\tau^i}$ for $i = 1, \ldots, n$, then for any $q = \sum_{i=1}^n \pi^i p^i \in \mathcal{C}$ we have by repeated application of ratio substitution that

$$q = \sum_{i=1}^n \pi^i p^i \sim \frac{\pi^1 \tau^1 \zeta_{\alpha^1} + \sum_{i=2}^n \pi^i p^i}{\pi^1 \tau^1 + \sum_{i=2}^n \pi^i} \sim \ldots \sim \frac{\sum_{i=1}^n \pi^i \tau^i \zeta_{\alpha^i}}{\sum_{i=1}^n \pi^i \tau^i} = \zeta_{\frac{\sum_{i=1}^n \pi^i \tau^i \alpha^i}{\sum_{i=1}^n \pi^i \tau^i}} = \zeta_{U(q)}$$

$$\beta q + (1-\beta) r \sim \frac{\beta[\sum_{i=1}^n \pi^i \tau^i] \zeta_{\frac{\sum_{i=1}^n \pi^i \tau^i \alpha^i}{\sum_{i=1}^n \pi^i \tau^i}} + (1-\beta) r}{\beta[\sum_{i=1}^n \pi^i \tau^i] + (1-\beta)} = \frac{\beta W(q) \zeta_{U(q)} + (1-\beta) r}{\beta W(q) + (1-\beta) r}$$

Hence the utility function may be expressed as a weighted expected value.

$$U(q) = \frac{\sum_{i=1}^n \pi^i W(p^i) U(p^i)}{\sum_{i=1}^n \pi^i W(p^i)} \qquad W(q) = \sum_{i=1}^n \pi^i W(p^i)$$

If $\mathcal{C} = \Delta(X)$, we may take X as the basis and define $u : X \mapsto \mathbb{R}$ and $w : X \mapsto \mathbb{R}^+$ such that $u(x) = U(\delta_x)$ and $w(x) = W(\delta_x)$ for every $x \in X$. Then for every $p, q \in \Delta(X)$ we have that

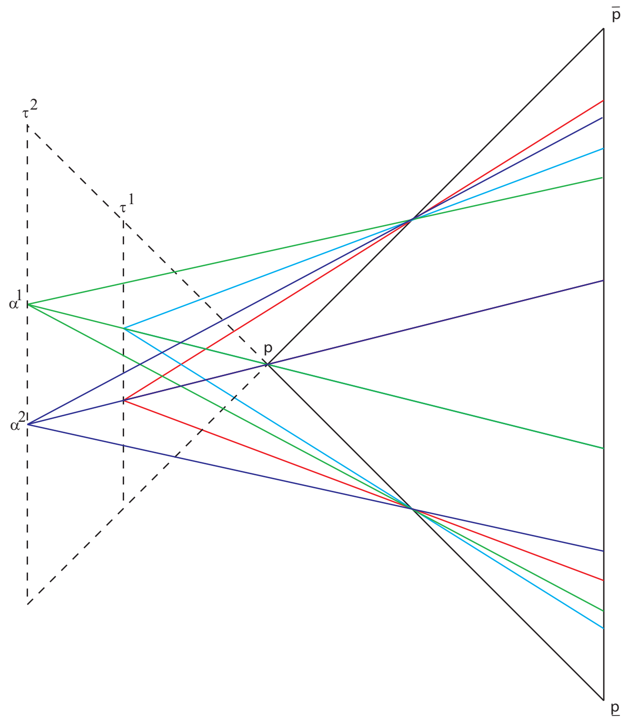
$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)} > \frac{\sum_{x \in X} q(x) w(x) u(x)}{\sum_{x \in X} q(x) w(x)}$$

Note that just as the multi-utility representation relied on the independence axiom to obtain its result, the weighted utility representation relies on the completeness axiom and the transitivity of the indifference relation in order to build up the utility function to take its weighted linear form.

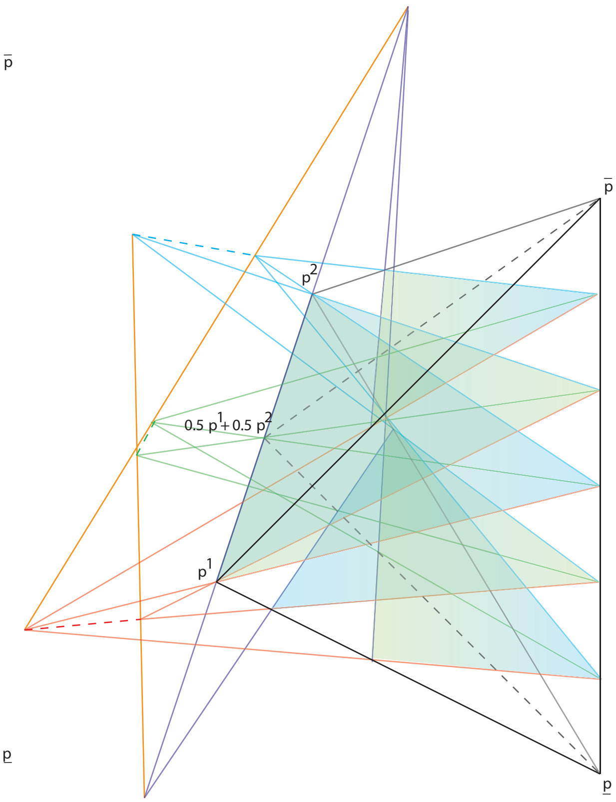
1.3 Conclusion

In the preceding sections we have considered models where preferences either violate completeness and satisfy independence, or vice versa, and considered how their conclusions might extend to a general setting where neither of these axioms hold. We have seen that in the multi-utility model, the indifference curves of expected utility are replaced by multiple sets of parallel incomparability curves that exhibit varying tastes, all of which must agree on a ranking of two lotteries for the decision maker to express a strict preference. On the other hand, in weighted utility model, the indifference curves are no longer parallel but instead are projected from a common source, so that the location of this point determines both the decision maker's attitude towards and perception of risk.

The road map to modeling preferences that satisfy neither completeness nor independence is now clear, as the indifference map should consist of multiple source points projecting multiple sets of incomparability curves that each may fan either in or out to varying degrees. As demonstrated in Figure 1.20a, the locations of these source points may correspond to different values of either α or τ, and thus may represent either multiple utility functions or multiple weight functions, corresponding to incompleteness in either tastes or perceptions. In higher dimensions, as in Figure 1.20b, we see that the structure of preferences must be such that the upper and lower contour sets are convex cones but not necessarily half spaces, as they would be under completeness, and that their shape varies moving around the space of mixtures, unlike in models where independence holds, so that the degree of incompleteness is also variable. Thus, if $\mathcal{C} = \Delta(X)$ there is a set $\tilde{\mathcal{V}}$ consisting of pairs (u, w) of utility $u : X \mapsto \mathbb{R}$



(a) Two Dimensions



(b) Three Dimensions

Fig. 1.20: Multiple Weighted Expected Utility

and weight $w : X \mapsto \mathbb{R}^+$ functions such that

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)} \qquad \forall; (u, w) \in \tilde{\mathcal{V}}$$

Note that the multiple weighted expected utility representation above admits as special cases the multi-utility models of Dubra, Maccheroni, and Ok [10] and Galaabaatar and Karni (2012) [13] if $w(\cdot)$ is a constant function for every $(u, w) \in \tilde{\mathcal{V}}$, or the weighted utility model of Chew and MacCrimmon (1979) [7] if $\tilde{\mathcal{V}}$ is a singleton, reducing to standard expected utility if both hold.

The next chapter will provide a formal proof for the representation theorem of the multiple weighted expected utility model, and the final chapter will explore some special cases of the general model, specifically those where the set of decision criteria consists of a multiple utilities paired with a single weight function $\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \times \{w\}$, or multiple weighting schemes paired with a single utility $\tilde{\mathcal{V}} = \{u\} \times \tilde{\mathcal{W}}$, so that incompleteness is restricted to either tastes or perception alone.

2. WEIGHTED EXPECTED UTILITY WITHOUT THE COMPLETENESS AXIOM

2.1 Overview

In the preceding chapter, we examined various models of choice under risk for preferences that violate either the completeness or independence axioms and considered how their results might be extended to a general framework where neither of these assumptions holds. We will now present a formal characterization of such a model, using the weighted expected utility setup of Chew and MacCrimmon (1979) [7] and relaxing the assumption of negative transitivity that ensures completeness. Recall the previously established setup, where $\succ$ is a strict preference relation over a mixture space $\mathcal{C}$, that obeys the following assumptions.

Axiom 1 (Strict Partial Order) For every $p, q \in \mathcal{C}$, if $p \succ q$ then $\neg(p \prec q)$ and for every $p, q, r \in \mathcal{C}$, if $p \succ q$ and $q \succ r$ then $p \succ r$.

Axiom 2 (Archimedean) For every $p, q, r \in \mathcal{C}$ if $p \succ r$ then there is $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha)q \succ r$ and if $q \prec r$ then there is $\beta \in (0,1)$ such that $\beta p + (1-\beta)q \prec r$.

Axiom 3 (Mixture Dominance) For every $p, q, r \in \mathcal{C}$ and $\alpha \in (0,1)$, if $p \succ r$ and $q \succ r$ then $\alpha p + (1-\alpha)q \succ r$, and if $p \prec r$ and $q \prec r$ then $\alpha p + (1-\alpha)q \prec r$.

Axiom 4 (Betweenness) For every $p, q \in \mathcal{C}$ and $\alpha \in (0,1)$, $p \succ q$ if and only if $p \succ \alpha p + (1-\alpha)q \succ q$.

Axiom 5 (Weak Substitution) For every $p, q \in \mathcal{C}$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that for every $r \in \mathcal{C}$, $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$.

The objective of this chapter will be to show that a preference relation satisfying these axioms has a multiple weighted expected utility representation. That is, $\succ$ is given by the agreement of a set of decision criteria, each of which in turn has a representation by a weighted linear utility function paired with a linear weight function.

Definition (Weighted Linearity) A weighted linear pair consists of a utility function $U : \mathcal{C} \mapsto \mathbb{R}$ and $W : \mathcal{C} \mapsto \mathbb{R}^+$ such that for every $p, q \in \mathcal{C}$ and $\pi \in [0,1]$, we have that

$$U(\pi p + (1-\pi)q) = \frac{\pi W(p)U(p) + (1-\pi)W(q)U(q)}{\pi W(p) + (1-\pi)W(q)}$$
$$W(\pi p + (1-\pi)q) = \pi W(p) + (1-\pi)W(q)$$

Theorem 1 A preference relation $\succ$ over a convex subset $\mathcal{C}$ of a finite dimensional linear space $\mathcal{L}$ is bounded and satisfies Axioms 1-5 if and only if there is a set $\mathcal{V}$ of weighted linear pairs such that for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow U(p) > U(q) \quad \forall\, (U, W) \in \mathcal{V}$$

If the space of mixtures is a set of lotteries $\Delta(X)$ over a finite set of prizes X, then every weighted linear pair (U, W) defines utility and weight functions over outcomes $u : X \mapsto \mathbb{R}$ and $w : X \mapsto \mathbb{R}^+$ by setting $u(x) = U(\delta_x)$ and $w(x) = W(\delta_x)$ for every $x \in X$, so that for every $p = \sum_{x \in X} p(x)\delta_x \in \Delta(X)$ we have

$$U(p) = \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} \qquad W(p) = \sum_{x \in X} p(x)w(x)$$

An immediate corollary to Theorem 1 gives the representation of a preference relation over $\Delta(X)$ satisfying the enumerated axioms, characterizing each of the decision criteria $(U, W) \in \mathcal{V}$ as a weighted expected value.

Corollary to Theorem 1 A preference relation $\succ$ over $\Delta(X)$ is bounded and satisfies Axioms 1-5 if and only if there is a set $\tilde{\mathcal{V}}$ of utility-weight pairs such that for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)} \quad \forall\ (u, w) \in \tilde{\mathcal{V}}$$

Observe that the representation above admits the models considered in the preceding chapter as special cases, taking the form of multi-utility whenever each weight function is constant, weighted utility whenever the set of decision criteria is a singleton, and expected utility if both of these are true. In general $\tilde{\mathcal{V}}$ may take a number of forms, possibly consisting of a multiple utility functions $\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \times \{w\}$, multiple weight functions $\tilde{\mathcal{V}} = \{u\} \times \tilde{\mathcal{W}}$, or both $\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \times \tilde{\mathcal{W}}$.

2.2 Model

To obtain our result, we will first consider the construction of a multiple weighted expected utility representation over the restriction of $\succ$ to some plane $\mathcal{L}(p)$, defined as before as the space spanned by the vectors $\overline{p} - p$ and $\underline{p} - p$. The overall representation is then established by connecting the indifference maps over the individual $\mathcal{L}(p)$ to one another, matching the decision criteria defined on these planes to generate utilities over the entirety of $\mathcal{C}$.

2.2.1 Two-Dimensional Model

Fix some $p \in \mathcal{C}$ and consider the restriction of the preference relation $\succ$ to $\mathcal{L}(p)$. Since $\overline{p} \succ p \succ \underline{p}$, there exists a range of mixtures of the best and worst elements to which p is incomparable, $p \asymp \alpha\overline{p} + (1-\alpha)\underline{p} \equiv \zeta_\alpha$. The following lemma shows that this range is a closed interval.

Lemma 4 (Solvability) If $\succ$ satisfies Axioms 1-4, then for every $p, q, r \in \mathcal{C}$ such that $p \succ r \succ q$ there are $\underline{\alpha}, \overline{\alpha} \in [0,1]$ such that $r \asymp \alpha p + (1-\alpha)q$ if and only if $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Thus the set of mixtures ζ_α to which p is incomparable gives the range of normalized utility values which p may take.

$$A(p) = \{\alpha \in [0,1] : p \asymp \zeta_\alpha\} = [\underline{\alpha}(p), \overline{\alpha}(p)]$$

Recall that under independence a multi-utility representation could be defined by a pair of functions corresponding to $\overline{\alpha}(p)$ and $\underline{\alpha}(p)$, corresponding to the most risk averse and risk loving of the decision criteria, respectively. While this is not possible here, we will show that the elements of $A(p)$ can still generate a representation over the entirety of $\mathcal{L}(p)$. For any incomparable pair $p \asymp q$, denote the set of odds ratios $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$ for β, γ satisfying weak substitution as

$$T(p,q) = \left\{\tau \geq 0 : \beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)} \; \forall \beta \in (0,1), r \in \mathcal{C}\right\}$$

Note that $T(p,q)$ was a singleton under completeness but need not be so here, and that each $\tau \in T(p,q)$ represents the weight of p relative to q according to some scheme that attaches disparate levels of importance to the various lotteries as components of mixtures. Therefore, if $\alpha \in A(p)$ and $\tau \in T(p, \zeta_\alpha)$, then the pair (α, τ) defines a utility-weight pair associated with p. Define the set of all such pairs for any $p \in \mathcal{C}$ as

$$\Phi(p) = \left\{ (\alpha, \tau) : \beta p + (1-\beta) r \asymp \frac{\beta \tau \zeta_\alpha + (1-\beta) r}{\beta \tau + (1-\beta)} \ \forall \ \beta \in (0,1), r \in \mathcal{C} \right\}$$

Note that as $\Phi(\overline{p}) = \{(1,1)\}$ and $\Phi(\underline{p}) = \{(1,0)\}$, the utilities and weights we will define are normalized to set $U(\overline{p}) = 1$, $U(\underline{p}) = 0$, and $W(\overline{p}) = W(\underline{p}) = 1$.

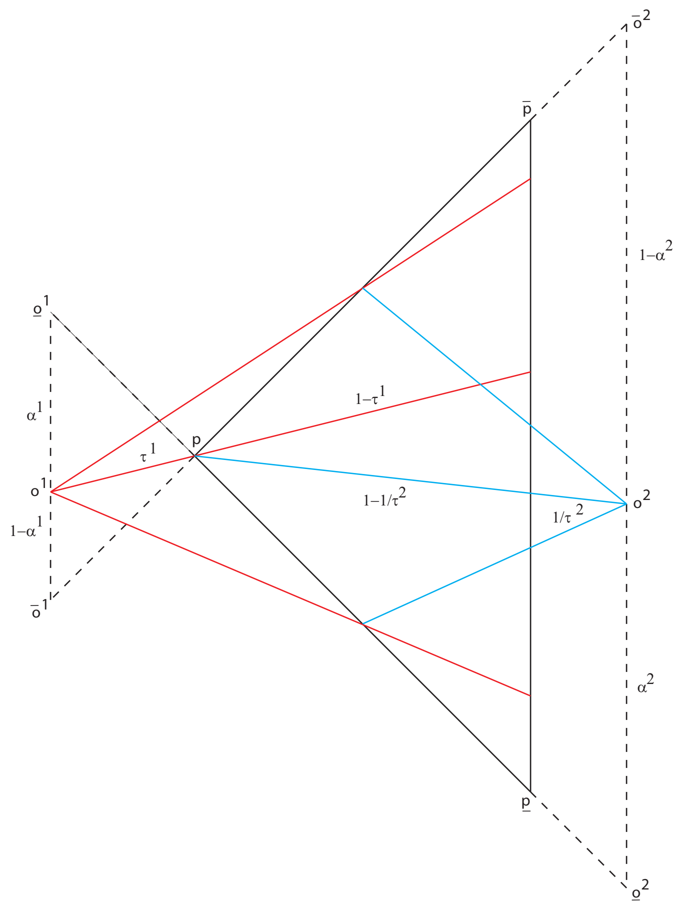


Fig. 2.1: Locating the Source

Each $\phi = (\alpha, \tau) \in \Phi(p)$ defines a set of incomparability curves over $\mathcal{L}(p)$ converging at a source point $o^\phi \in \mathcal{L}(p)$ located at

$$o^\phi = \frac{p - \tau \zeta_\alpha}{1-\tau} = \frac{1}{1-\tau} p - \frac{\tau \alpha}{1-\tau} \overline{p} - \frac{\tau(1-\alpha)}{1-\tau} \underline{p}$$

As shown in Figure 2.1 the source point o^ϕ is located in the Machina (1982) [22] triangle diagram by drawing a similar triangle intersecting the simplex at p for $\tau < 1$ or extending the simplex for $\tau > 1$, and extending the line between p and ζ_α to intersect the opposite edge. The pair (α, τ) defines a useful coordinate system which we will take advantage of throughout the course of this analysis. Since $p = \tau\zeta_\alpha + (1-\tau)o^\phi$, the odds ratio determines the distance of the source point from the simplex itself $\tau = \frac{\|p-o^\phi\|}{\|\zeta_\alpha - o^\phi\|}$. As the fanning effect of the incomparability curves is magnified as the source is moved closer to the simplex, the deviation of the odds ratio from unity determines the degree to which the decision maker's perception of risk is distorted from what the independence axiom would prescribe, and hence τ gives the "horizontal" location of the source point. Now letting $\overline{o}^\tau = \frac{p-\tau\overline{p}}{1-\tau}$ and $\underline{o}^\tau = \frac{p-\tau\underline{p}}{1-\tau}$, note that the vector $\overline{o}^\tau - \underline{o}^\tau$ is parallel to $\overline{p} - \underline{p}$, and the source point is located at the intersection of the line connecting $\overline{o}^\tau$ and $\underline{o}^\tau$ and the incomparability curve connecting p with ζ_α, so that $o^\phi = \alpha\overline{o}^\tau + (1-\alpha)\underline{o}^\tau$, and α gives the "vertical" location of the source point.

The critical observation here is that each source point, and consequently the incomparability curves it projects, is defined irrespective of the actual lottery p used to locate it, and therefore $\Phi(p)$ uniquely determines $\Phi(q)$ for any $q \in \mathcal{L}(p)$.

Lemma 5 If $\succ$ satisfies Axiom 5, then for every $p \in \mathcal{C}$, $(\alpha, \tau) \in \Phi(p)$ if and only if for every $q \in \mathcal{L}(p)$ there is $(\alpha', \tau') \in \Phi(q)$ such that $\frac{p-\tau\zeta_\alpha}{1-\tau} = \frac{q-\tau'\zeta_{\alpha'}}{1-\tau'}$.

Therefore, from each $\phi = (\alpha, \tau) \in \Phi(p)$ we can define utility $U^\phi : \mathcal{L}(p) \mapsto \mathbb{R}$ and weight $W^\phi : \mathcal{L}(p) \mapsto \mathbb{R}^+$ functions by setting for every $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$,

$$U^\phi(q) = \frac{\lambda\tau\alpha + (1-\lambda)\theta}{\lambda\tau + (1-\lambda)} \qquad W^\phi(q) = \lambda\tau + (1-\lambda)$$

It is easily verified that (U^ϕ, W^ϕ) is a weighted linear pair, since if $q^i = \lambda^i p + (1-\lambda^i)\zeta_{\theta^i}$

for $i = 1, 2$ and $\pi \in [0, 1]$ we have

$$\begin{aligned}
\pi q^1 + (1-\pi)q^2 &= \pi[\lambda^1 p + (1-\lambda^1)\zeta_{\theta^1}] + (1-\pi)[\lambda^2 p + (1-\lambda^2)\zeta_{\theta^2}] \\
&= [\pi\lambda^1 + (1-\pi)\lambda^2]p + [\pi(1-\lambda^1)\zeta_{\theta^1} + (1-\pi)(1-\lambda^2)\zeta_{\theta^2}] \\
U^\phi(\pi q^1 + (1-\pi)q^2) &= \frac{[\pi\lambda^1 + (1-\pi)\lambda^2]\tau\alpha + [\pi(1-\lambda^1)\theta^1 + (1-\pi)(1-\lambda^2)\theta^2]}{[\pi\lambda^1 + (1-\pi)\lambda^2]\tau + [\pi(1-\lambda^1) + (1-\pi)(1-\lambda^2)]} \\
&= \frac{\pi[\lambda^1\tau\alpha + (1-\lambda^1)\theta^1] + (1-\pi)[\lambda^2\tau + (1-\lambda^2)\theta^2]}{\pi[\lambda^1\tau + (1-\lambda^1)] + (1-\pi)[\lambda^2\tau + (1-\lambda^2)]} \\
&= \frac{\pi W^\phi(q^1)U^\phi(q^1) + (1-\pi)W^\phi(q^2)U^\phi(q^2)}{\pi W^\phi(q^1) + (1-\pi)W^\phi(q^2)} \\
W^\phi(\pi q^1 + (1-\pi)q^2) &= [\pi\lambda^1 + (1-\pi)\lambda^2]\tau + [\pi(1-\lambda^1) + (1-\pi)(1-\lambda^2)] \\
&= \pi[\lambda^1\tau + (1-\lambda^1)] + (1-\pi)[\lambda^2\tau + (1-\lambda^2)] \\
&= \pi W^\phi(q^1) + (1-\pi)W^\phi(q^2)
\end{aligned}$$

By Lemma 5 we have that $\phi' \in \Phi(q)$ if and only if there is $\phi \in \Phi(p)$ such that $\phi' = (U^\phi(q), W^\phi(q))$. Repeat this for each $\phi \in \Phi(p)$ and collect these functions in $\mathcal{V}(p) = \{(U^\phi, W^\phi) : \phi \in \Phi(p)\}$.

Figure 2.2 shows the construction of $\mathcal{V}(p)$, as each $\phi \in \Phi(p)$ defines a set of incomparability curves that converge at the source point o^ϕ, and defines a preference relation $\succ^\phi$ which has a a weighted expected utility representation by a pair of functions (U^ϕ, W^ϕ). The actual preferences are defined by the agreement of each of these, so that $p \succ q$ if and only if $p \succ^\phi q$ for every $\phi \in \Phi(p)$. Hence the overall indifference map is therefore constructed by superimposing the incomparability curves generated by each $\phi \in \Phi(p)$, yielding upper and lower contour sets which do not necessarily maintain their shape throughout the space of mixtures. In the example given, we have that $U(p) < U(q)$ but $U'(p) > U'(q)$, so that the two decision criteria disagree on how to rank p and q and according they lie in each other's incomparability sets so that the decision maker cannot rank them, and corresponding they must lie within each

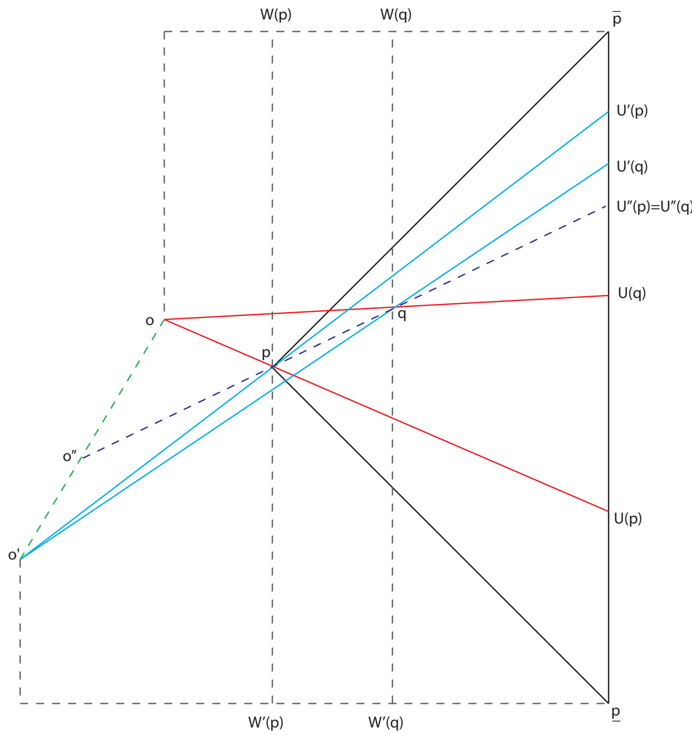


Fig. 2.2: Interpretation of Multiple Weighted Utility

other's incomparability sets. This also indicates how the collection of decision criteria $\mathcal{V}(p)$ represents $\succ$. Any point o'' on the line connecting two sources must itself be a source, as the incomparability curve originating from it through any $q \in \mathcal{L}(p)$ must lie between the corresponding curves originating from o and o'. Conversely therefore, given any pair of incomparable lotteries such as p and q, there must exist a source point o'' on the extended line between them, so that further extending this line to intersect the line connecting $\overline{p}$ and $\underline{p}$ shows that the utility function generated by o'' assigns the same value to both p and q. The following lemma formalizes this result.

Lemma 6 If $\succ$ satisfies Axioms 1-5, then for every $q^1, q^2 \in \mathcal{L}(p)$, $q^1 \asymp q^2$ if and only if $U(q^1) = U(q^2)$ for some $(U, W) \in \mathcal{V}(p)$.

The proof that $\mathcal{V}(p)$ represents $\succ$ thus follows directly from this, since if q^2 does not lie on any of the incomparability curves intersecting q^1, it must either lie above or below all of them, implying that q^2 is either strictly better or strictly worse than q^1.

Lemma 7 If $\succ$ satisfies Axioms 1-5, then for every $q^1, q^2 \in \mathcal{L}(p)$, $q^1 \succ q^2$ if and only if $U(q^1) > U(q^2)$ for every $(U, W) \in \mathcal{V}(p)$.

Thus, the restriction of the preference relation $\succ$ to any two-dimensional slice $\mathcal{L}(p)$ of the simplex has a multiple weighted expected utility representation, found by taking the set of utility-weight pairs $\Phi(p)$ that satisfy both solvability and ratio substitution. Each such pair $\phi \in \Phi(p)$ defines a source point that projects a set of incomparability curves and hence generates a weighted linear pair (U^ϕ, W^ϕ) representing a complete preference relation $\succ^\phi$ where $q^1 \succ^\phi q^2$ if and only if $U^\phi(q^1) > U^\phi(q^2)$. Since $q^1 \succ q^2$ if and only if $q^1 \succ^\phi q^2$ for every $\phi \in \Phi(p)$, collecting all the utility-weight pairs constructed from $\Phi(p)$ defines a set $\mathcal{V}(p)$ that forms the desired representation of $\succ$.

2.2.2 General Model

We now consider the problem of obtaining a multiple weighted expected utility representation for $\succ$ over the entire space $\mathcal{C}$. Let $U : \mathcal{C} \mapsto \mathbb{R}$ and $W : \mathcal{C} \mapsto \mathbb{R}^+$ denote utility and weight functions over $\mathcal{C}$ such that (U, W) is weighted linear, $U(p^1) = U(p^2)$ implies $p^1 \asymp p^2$, and $p^1 \succ p^2$ implies $U(p^1) > U(p^2)$. This utility function is compatible with the preference $\succ$ in the sense of Aumann (1962) [2], but does alone represent it. This implies for every $p \in \mathcal{C}$ that $(U(p), W(p)) \in \Phi(p)$ and hence that there is $(U^p, W^p) \in \mathcal{V}(p)$ such that $(U^p(q), W^p(q)) = (U(q), W(q))$ for every $q \in \mathcal{L}(p)$. Hence, similar to multi-utility models under independence, the problem of finding a representation for $\succ$ over $\mathcal{C}$ reduces to matching the utility-weight pairs over the individual $\mathcal{V}(p)$ to one another, constructing a set $\mathcal{V}$ of weighted linear pairs such that $(U, W) \in \mathcal{V}$ if and only if the restriction of (U, W) to $\mathcal{L}(p)$ is an element of $\mathcal{V}(p)$ for every $p \in \mathcal{C}$.

Suppose we have $p^1, p^2 \in \mathcal{C}$ for which $p^2 \notin \mathcal{L}(p^1)$ and $\phi^i = (\alpha^i, \tau^i) \in \Phi(p^i)$ for $i = 1, 2$. For the sake of brevity, given $\pi \in [0, 1]$ let $p^\pi \equiv \pi p^1 + (1 - \pi)p^2$ and denote the weighted linear mixture of the utility-weight pairs as

$$\phi^\pi \equiv \pi\phi^1 \oplus (1-\pi)\phi^2 = \left(\frac{\pi\tau^1\alpha^1 + (1-\pi)\tau^2\alpha^2}{\pi\tau^1 + (1-\pi)\tau^2}, \pi\tau^1 + (1-\pi)\tau^2\right)$$

If $\succ$ has a multiple weighted utility representation over $\mathcal{C}$, then for every pair $(U, W) \in \mathcal{V}$ we have that $\phi^i = (U(p^i), W(p^i)) \in \Phi(p^i)$ for $i = 1, 2$. By the weighted linearity of (U, W), for every π we must have that $\phi^\pi = \pi\phi^1 \oplus (1-\pi)\phi^2 \in \Phi(p^\pi)$ and hence that $U(q) = U^{\phi^\pi}(q)$ for every $q \in \mathcal{L}(p^\pi)$. Furthermore, letting $o^i = \frac{p^i - \tau^i\zeta_{\alpha^i}}{1-\tau^i}$ for $i = 1, 2$, we have for every π that there is a source point located at

$$\begin{aligned} o^\pi &\equiv \frac{[\pi p^1 + (1-\pi)p^2] - [\pi\tau^1 + (1-\pi)\tau^2]\zeta_{\frac{\pi\tau^1\alpha^1 + (1-\pi)\tau^2\alpha^2}{\pi\tau^1 + (1-\pi)\tau^2}}}{1[\pi\tau^1 + (1-\pi)\tau^2]} \\ &= \frac{\pi(p^1 - \tau^1\zeta_{\alpha^1}) + (1-\pi^1)(p^2 - \tau^2\zeta_{\alpha^2})}{\pi(1-\tau^1) + (1-\pi^1)(1-\tau^2)} = \frac{\pi(1-\tau^1)o^1 + (1-\pi)(1-\tau^2)o^2}{\pi(1-\tau^1) + (1-\pi)(1-\tau^2)} \end{aligned}$$

Hence each $(U, W) \in \mathcal{V}$ corresponds to defining a source line $O = \{o^\pi : \pi \in [0, 1]\}$ that projects a set of incomparability planes over the space $\bigcap_{\pi\in[0,1]} \mathcal{L}(p^\pi)$, though in fact by betweenness the incomparability plane and hence the source line may be infinitely extended to encompass any $\pi \in \mathbb{R}$. Note however that just as in models of incomplete preferences with independence, this matching cannot be done arbitrarily due to the intransitivity of the incomparability relation. Suppose we have $(\alpha^i, \tau^i) \in \Phi(p^i)$ for $i = 1, 2$, then even though we have by ratio substitution that $\pi p^1 + (1-\pi)p^2 \asymp \frac{\pi\tau^1\zeta_{\alpha^1} + (1-\pi)p^2}{\pi\tau^1 + (1-\pi)} \asymp \frac{\pi\tau^1\zeta_{\alpha^1} + (1-\pi)\tau^2\zeta_{\alpha^2}}{\pi\tau^1 + (1-\pi)\tau^2}$, we may have that $\pi p^1 + (1-\pi)p^2 \succ \frac{\pi\tau^1\zeta_{\alpha^1} + (1-\pi)\tau^2\zeta_{\alpha^2}}{\pi\tau^1 + (1-\pi)\tau^2}$. For example suppose that $\Phi(p^1) = [\frac{1}{6}, \frac{1}{2}] \times \{\frac{1}{2}\}$ and $\Phi(p^2) = [\frac{1}{2}, \frac{5}{6}] \times \{\frac{1}{3}\}$, so that the incomparability curves through p^1 and p^2 are respectively projected from the sets of source points $S^1 = \{2p^1 - \zeta_{\alpha^1} : \alpha^1 \in [\frac{1}{6}, \frac{1}{2}]\} \subseteq \mathcal{L}(p^1)$ and $S^2 = \{\frac{3}{2}p^2 - \frac{1}{2}\zeta_{\alpha^2} : \alpha^2 \in [\frac{1}{2}, \frac{5}{6}]\} \subseteq \mathcal{L}(p^2)$. Suppose that $\mathcal{V}$ is characterized by a pair of decision criteria

$\{(U,W),(U',W')\}$ such that for $\pi \in [0,1]$ we have

$$(U(p^\pi), W(p^\pi)) = \left(\frac{1}{2}, \frac{1}{3} + \frac{1}{6}\pi\right)$$
$$(U'(p^\pi), W'(p^\pi)) = \left(\frac{\frac{5}{18} - \frac{7}{36}\pi}{\frac{1}{3} + \frac{1}{6}\pi}, \frac{1}{3} + \frac{1}{6}\pi\right)$$

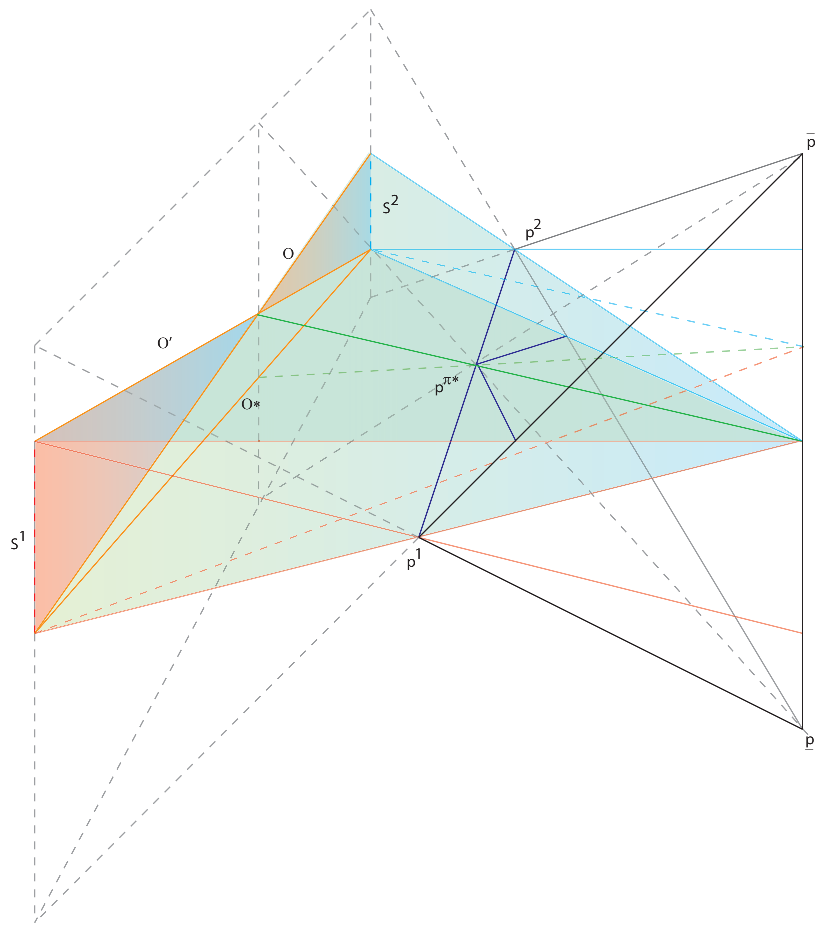


Fig. 2.3: Representation by Two Weighted Linear Utilities

As shown in Figure 2.3, both of the decision criteria represent identical weight schemes, but assign a range of utility values to every p^π. Observe that the source lines O and O' respectively representing (U,W) and (U',W') are defined by connecting source points in S^1 and S^2, and furthermore every point lying on either line is itself a source, so that both lines project sets of incomparability planes. Note that for $\pi^* = \frac{2}{5}$, the utilities exactly coincide so that $U(p^{\pi^*}) = U'(p^{\pi^*}) = \frac{1}{2}$, and $\Phi(p^{\pi^*}) = \{\frac{1}{2}\} \times \{\frac{2}{5}\}$, so that the lone source point $o^{\pi^*} = \frac{5}{3}p^{\pi^*} - \frac{2}{3}\zeta_{\frac{1}{2}} \in \mathcal{L}(p^{\pi^*})$ is given by the intersection of the source lines O and O'. Since $p^{\pi^*} \sim \zeta_{\frac{1}{2}}$, its upper and lower contour sets are wedge-shaped, similar to how they would be if preferences were represented by a pair of linear utilities. Note however that even though the line O^* also connects points in S^1 and S^2, it is not a source line, as the plane it would project through p^{π^*} intersects the best-worst line at $\zeta_{\frac{2}{3}} \succ p^{\pi^*}$, and hence is not an incomparability plane.

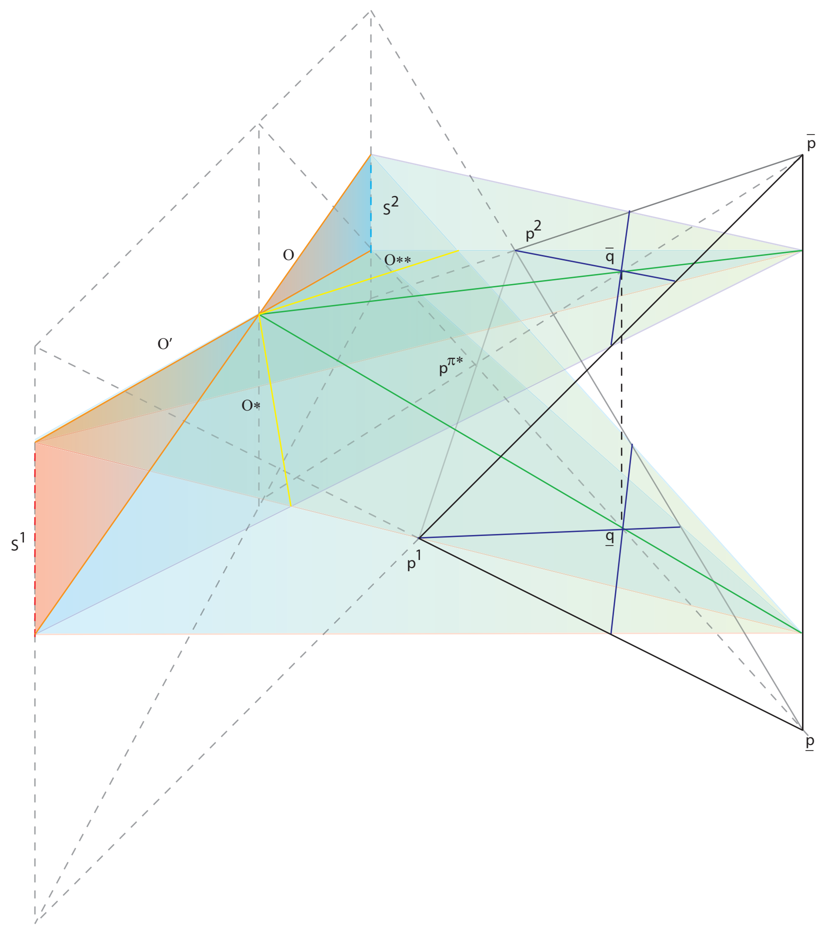


Fig. 2.4: Locating the Source Lines

Recall that in the preceding chapter, we showed that constructing a multi-utility representation relied on applying the independence axiom, while constructing a weighted utility representation used the completeness axiom. Figure 2.4 shows that since neither of those properties holds here, we cannot use the same techniques to find our representation. In the models of Dubra, Maccheroni, and Ok (2004) [10] and Galaabataar and Karni (2012) [13], the problem of matching utilities was resolved by observing that the shape of the upper contour set $B(p)$ was identical at every $p \in \mathcal{C}$ as a result of the independence axiom, and hence taking the set of supporting hyperplanes of any $B(p)$ defined the set of utilities. Note here however that letting $\overline{q} = \beta p^{\pi^*} + (1-\beta)\overline{p}$ and $\underline{q} = \beta p^{\pi^*} + (1-\beta)\underline{p}$, no supporting hyperplane of $B(\overline{q})$ is also a supporting hyperplane of $B(\underline{q})$. Indeed, there is no easily defined analogue of the domination cone $\mathcal{D} = \{\lambda(p-q) : p \succ q, \lambda \geq 0\}$ within this setup, as Axiom 5 cannot be equivalently expressed in terms of strict preference the same way that the independence axiom can. On the other hand, in the weighted utility model of Chew and MacCrimmon (1979) [7], taking the intersection of any pair of indifference planes gave the unique source line, and any other plane projected from this line would also be an indifference plane. Observe that this result does not translate here either, as the lines O^* and O^{**} lie at the intersection of two incomparability planes, but are not composed of source points and hence do not themselves project incomparability planes.

In order to begin the process of defining a representation here, recall a property established in the preceding section that the source points are defined irrespective of the lotteries used to locate them. Thus given any $(\alpha,\tau) \in \Phi(p)$ that defines a source point $o = \frac{p-\tau\zeta_\alpha}{1-\tau}$, by Lemma 5 we can for any (α',τ') find some $p' \in \mathcal{L}(p)$ such that $(\alpha',\tau') \in \Phi(p')$ by setting

$$p' = \tau'\zeta_{\alpha'} + (1-\tau')o = \tau'\zeta_{\alpha'} + (1-\tau')\frac{p-\tau\zeta_\alpha}{1-\tau} = \frac{1-\tau'}{1-\tau}p + \frac{\tau'-\tau}{1-\tau}\zeta_{\frac{\tau'(1-\tau)\alpha'-\tau(1-\tau')\alpha}{\tau'(1-\tau)-\tau(1-\tau')}}$$

Therefore, given any basis $\{p^1,\ldots,p^n\}$ of $\mathcal{C}$ and $(\alpha^i,\tau^i) \in \Phi(p^i)$ for $i = 1,\ldots,n$, we can instead fix (α,τ) and find $q^i \in \mathcal{L}(p^i)$ such that $(\alpha,\tau) \in \Phi(q^i)$ for every i. If we can find a source hyperplane such that $(\alpha,\tau) \in \Phi(\sum_{i=1}^n \pi^i q^i)$ for every $(\pi^1,\ldots,\pi^n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n \pi^i = 1$, then we can define a weighted linear pair (U,W) over $\mathcal{C}$ by setting $(U(q), W(q)) = (\alpha, \tau)$ for every $q = \sum_{i=1}^{n} \pi^i q^i$, and applying Lemma 5 to define (U, W) over the plane $\mathcal{L}(q)$. Since $\{q^1, \ldots, q^n\}$ will span $\mathcal{C}$ as well, this defines (U, W) over the entirety of the mixture space. The set of decision criteria $\mathcal{V}$ may then be constructed by finding every such $\{q^1, \ldots, q^n\}$ that defines such a source hyperplane relative to the pair (α, τ). To establish the existence of a representation, we begin by showing the preliminary result that any pair of alternatives are incomparable if and only if the entire line between them shares a utility value in common.

Lemma 8 If $\succ$ satisfies Axioms 1-4, then for every $p^1, p^2 \in \mathcal{C}$, $p^1 \asymp p^2$ if and only if there is $\alpha \in [0, 1]$ such that $\pi p^1 + (1 - \pi)p^2 \asymp \zeta_\alpha$ for every $\pi \in \mathbb{R}$.

Lemma 8 shows that two lotteries are incomparable if and only if there is some incomparability plane on which they both lie, which can be extended to intersect some ζ_α that defines a utility value α such that $p^\pi \asymp \zeta_\alpha$ for every π. By ratio substitution there exists for every p^π a set of odds ratios $T(p^\pi, \zeta_\alpha)$ such that $o^\pi = \frac{p^\pi - \tau^\pi \zeta_\alpha}{1 - \tau^\pi}$ is a source point for every $\tau^\pi \in T(p^\pi, \zeta_\alpha)$. Note however that a common utility value does not alone ensure the existence of a source line, since this would additionally require that there are weights τ^1, τ^2 such that $\pi \tau^1 + (1 - \pi)\tau^2 \in T(p^\pi, \zeta_\alpha)$ for every π, thus defining a linear weight function and hence a weighted linear utility function.

As shown in Figure 2.5 for $i = 1, 2$ the incomparability curves intersecting p^i are projected from the source space $S^i = \left\{ o^i = \frac{p^i - \tau^i \zeta_{\alpha^i}}{1 - \tau^i} : (\alpha^i, \tau^i) \in \Phi(p^i) \right\} \subseteq \mathcal{L}(p^i)$, which are each triangular so that both p^1 and p^2 exhibit multiplicity of both utilities and weights. By Lemma 8 $p^1 \asymp p^2$ if and only if the entire line connecting them lies on an incomparability plane and shares the utility value α in common. Observe that every source point o^π that projects incomparability curves over $\mathcal{L}(p^\pi)$ lies on a source line O connecting some $o^1 \in S^1$ and $o^2 \in S^2$, but that not every line connecting points in S^1 and S^2 is a source line, as O^* would assign the utility value α^* to p^π even though $p^\pi \prec \zeta_{\alpha^*}$.

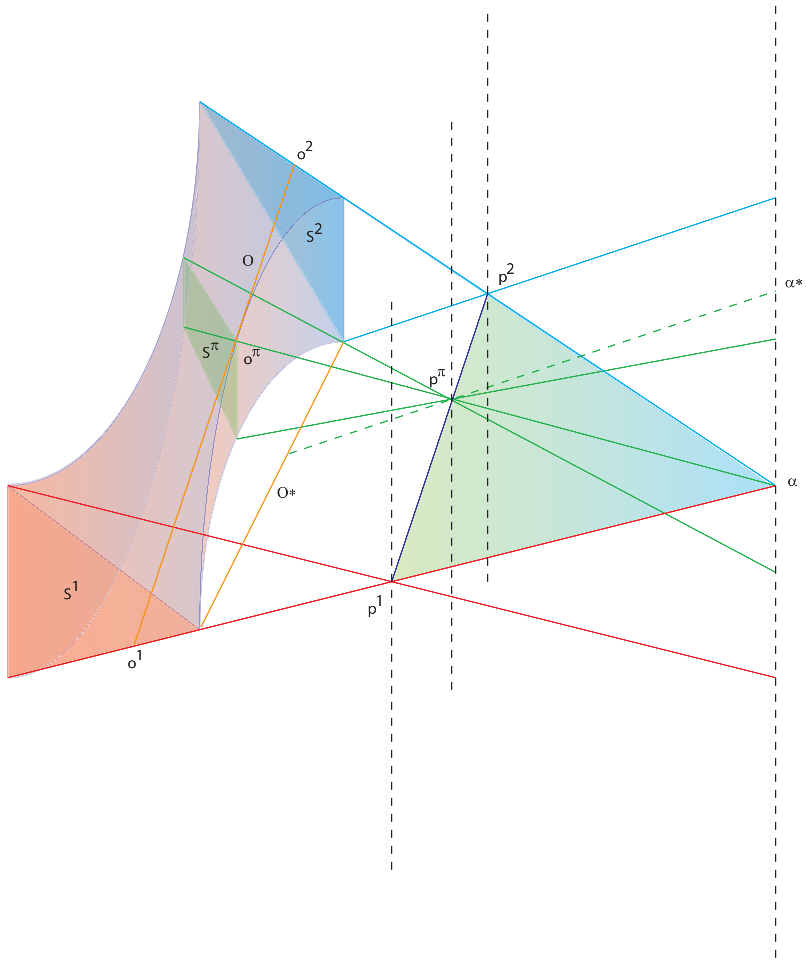


Fig. 2.5: Common Utility Value Defines an Incomparability Plane

As shown in Figure 2.6, for every π we have that $p^\pi \asymp \zeta_\alpha$, and hence by ratio substitution, for every odds ratio $\tau^\pi \in T(p^\pi, \zeta_\alpha)$, there is a source point $o^\pi = \frac{p^\pi - \tau^\pi \zeta_\alpha}{1-\tau^\pi}$ projecting a set of incomparability curves over $\mathcal{L}(p^\pi)$. The source line O is constructed by picking $\tau^i \in T(p^i, \zeta_\alpha)$ for $i = 1, 2$ such that $\tau^\pi \equiv \pi\tau^1 + (1-\pi)\tau^2 \in T(p^\pi, \zeta_\alpha)$ for every π. Thus we can define a linear weight function and hence a weighted linear utility function over $\bigcap_{\pi\in[0,1]} \mathcal{L}(p^\pi)$ by letting $(U(p^\pi), W(p^\pi)) = (\alpha, \tau^\pi)$ for every π. As with the utilities, these weight values cannot be matched arbitrarily, since O' is not a source line as it assigns a weight to p^π that lies outside of the range $T(p^\pi, \zeta_\alpha)$. In order to find the set of valid source lines, we apply Lemma 5 once again, reducing this problem to one of finding a pair of lotteries such that every linear combination

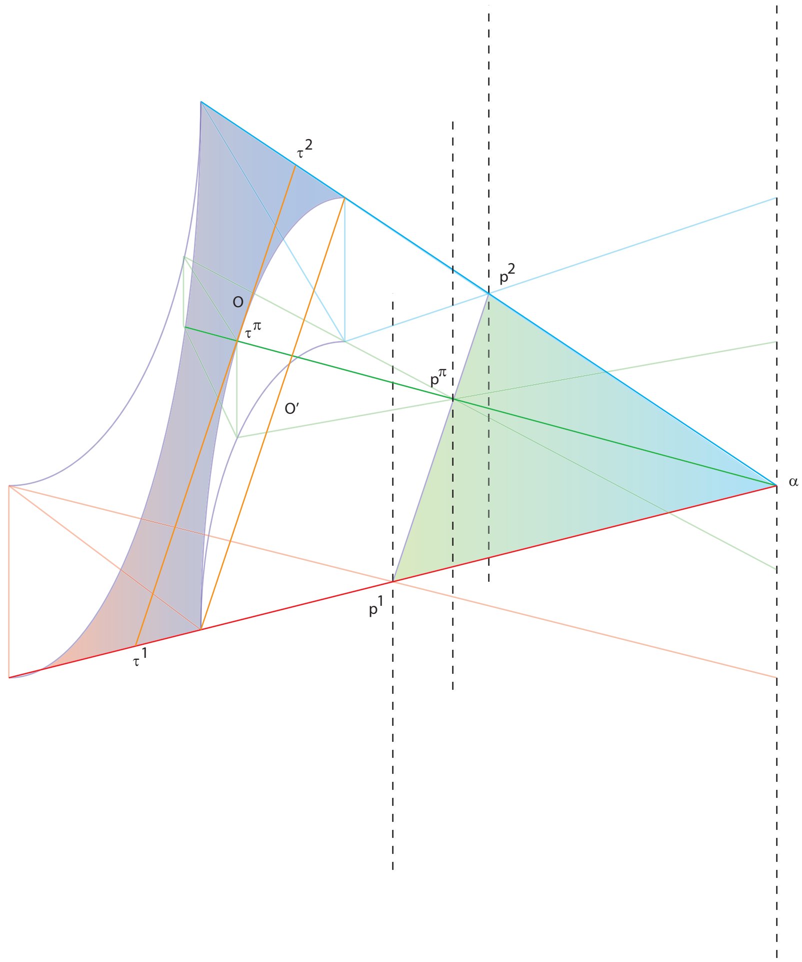


Fig. 2.6: Range of Weight Values for A Single Utility Value

of them shares a utility-weight pair (α, τ) in common.

Suppose that we have $p, p' \in \mathcal{C}$ such that $p \asymp p'$, and hence for some α we have that $\pi p + (1-\pi)p' \asymp \zeta_\alpha$ for every π. If a source line exists then there are $\tau \in T(p, \zeta_\alpha)$ and $\tau' \in T(p', \zeta_\alpha)$ such that $(\alpha, \pi\tau + (1-\pi)\tau') \in \Phi(\pi p + (1-\pi)p')$ for every π. Suppose $\tau' \neq \tau$, then by Lemma 5 we have that there is $q = \frac{1-\tau'}{1-\tau}p - \frac{\tau'-\tau}{1-\tau}\zeta_\alpha \in \mathcal{L}(p')$ such that

$(\alpha, \tau) \in \Phi(q)$. Furthermore, since we have for every $\pi \in [0,1]$ that

$$\begin{aligned}
\pi p + (1-\pi)q &= \pi p + (1-\pi)\left[\frac{1-\tau'}{1-\tau}p - \frac{\tau'-\tau}{1-\tau}\zeta_\alpha\right] \\
&= \frac{\pi(1-\tau)p + (1-\pi)(1-\tau')p' + (1-\pi)(\tau'-\tau)\zeta_\alpha}{1-\tau} \\
&\equiv \lambda'[\pi' p + (1-\pi')p'] + (1-\lambda')\zeta_{\theta'} \in \mathcal{L}(\pi' p + (1-\pi')p')
\end{aligned}$$

Since $(\alpha, \pi'\tau + (1-\pi')\tau') \in \Phi(\pi' p + (1-\pi')p')$, by another application of Lemma 5 we have that

$$\begin{aligned}
(\alpha, \lambda'[\pi'\tau + (1-\pi')\tau'] + (1-\lambda')) &= \left(\alpha, \frac{\pi(1-\tau)\tau + (1-\pi)(1-\tau')\tau' + (1-\pi)(\tau'-\tau)}{1-\tau}\right) \\
&= (\alpha, \tau) \in \Phi(\pi p + (1-\pi)q)
\end{aligned}$$

Hence a source line exists if and only if we can find $q \in \mathcal{L}(p')$ such that $(\alpha, \tau) \in \Phi(\pi p + (1-\pi)q)$ for every π. Note that given the interpretation of the odds ratio as a relative weight between a pair of incomparable alternatives, if p and q share both a utility and weight value in common, then they can be substituted for each other with an odds ratio of 1, so that $\beta p + (1-\beta)r \asymp \beta q + (1-\beta)r$ for every β and r. This implies that on any plane containing both p and q we can find a set of parallel incomparability curves.

For every $\delta \in \mathbb{R}$, let $p + \delta(\overline{p} - \underline{p})$ define a vertical shift from p in the direction $\overline{p} - \underline{p}$. Note that if $(\alpha, \tau) \in \Phi(p)$, then for every $\delta \in \mathbb{R}$,

$$o = \frac{p - \tau\zeta_\alpha}{1-\tau} = \frac{p + \delta(\overline{p} - \underline{p}) - \tau[\zeta_\alpha + \frac{\delta}{\tau}(\overline{p} - \underline{p})]}{1-\tau} = \frac{p + \delta(\overline{p} - \underline{p}) - \tau\zeta_{\alpha + \frac{\delta}{\tau}}}{1-\tau}$$

By Lemma 5, we have that $(\alpha, \tau) \in \Phi(p)$ if and only if $(\alpha + \frac{\delta}{\tau}, \tau) \in \Phi(p + \delta(\overline{p} - \underline{p}))$ for every $\delta \in \mathbb{R}$. The following lemma shows that if there exists a set of parallel incomparability curves connecting two such vertical lines defined this way, then any set of lotteries lying on such an incomparability curve shares a utility-weight pair in common.

Lemma 9 If $\succ$ satisfies Axioms 1-5, then for every $p^1, p^2 \in \mathcal{C}$, $p^1 + \delta(\overline{p} - \underline{p}) \asymp p^2 + \delta(\overline{p} - \underline{p})$ for every $\delta \in \mathbb{R}$ if and only if there is $\alpha \in [0, 1]$ and $\tau \geq 0$ such that $(\alpha, \tau) \in \Phi(\pi p^1 + (1 - \pi)p^2)$ for every $\pi \in \mathbb{R}$.

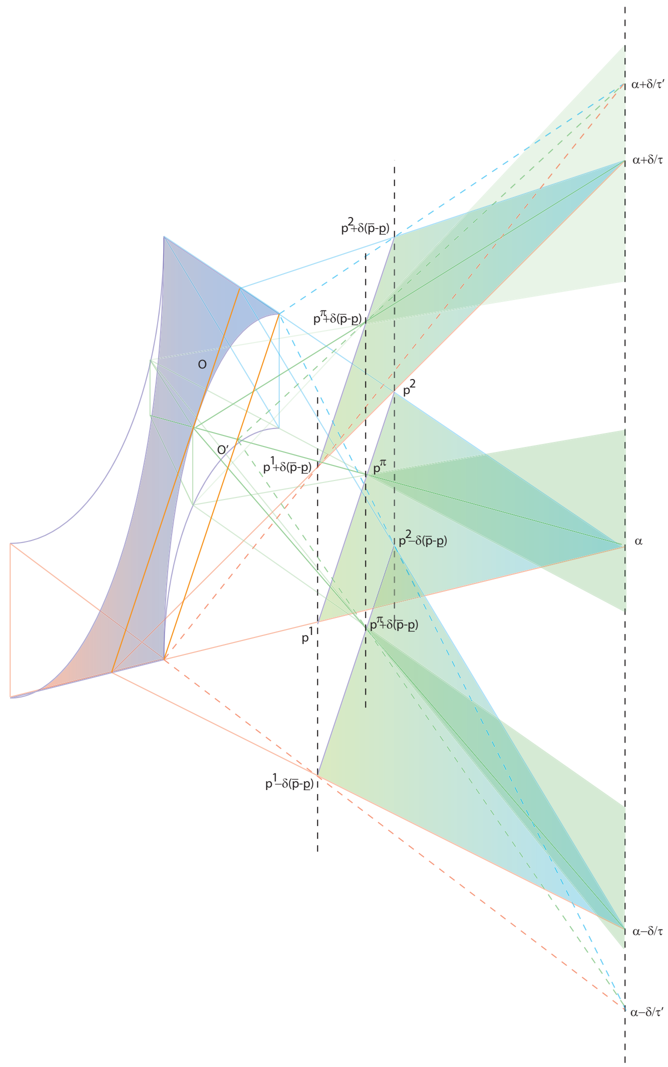


Fig. 2.7: Common Utility and Weight Values Define a Source Line

This lemma shows that there is a common utility-weight pair $(\alpha, \tau) \in \Phi(p^\pi)$ for every π if and only if for every δ there is a common utility value $\alpha + \frac{\delta}{\tau} \in A(p^\pi + \delta(\overline{p} - \underline{p}))$ for every π. As shown in Figure 2.7, this implies that there are a set of parallel incomparability curves connecting $p^1 + \delta(\overline{p} - \underline{p})$ to $p^2 + \delta(\overline{p} - \underline{p})$ for every $\delta \in \mathbb{R}$ if and only if there is a source line O that is itself parallel to the line connecting p^1 and p^2 projecting the incomparability planes on which these curves lie, establishing that $p^\pi + \delta(\overline{p} - \underline{p}) \asymp \zeta_{\alpha + \frac{\delta}{\tau}}$ for every π and δ. Note that here, a source line is defined only by the intersection of incomparability planes through $p^\pi + \delta(\overline{p} - \underline{p})$ for every $\delta \in \mathbb{R}$, unlike in weighted utility with completeness where the intersection of only two such planes was sufficient to define a source line. Observe that the line O' is also parallel to the line between p^1 and p^2 and hence represents another utility-weight pair (α, τ') with $\tau' < \tau$. Since $p^\pi \asymp \zeta_\alpha$ and $p^\pi + \delta(\overline{p} - \underline{p}) \asymp \zeta_{\alpha + \frac{\delta}{\tau'}}$, O' does indeed project at least a pair of incomparability planes, but since $p^\pi - \delta(\overline{p} - \underline{p}) \succ_{\alpha - \frac{\delta}{\tau'}}$, it cannot be a source line.

Therefore, we now have the condition that characterizes a source line O connecting source points in S^1 and S^2. The following lemma shows the existence of such a line for any $p^1, p^2 \in \mathcal{C}$, and furthermore establishes that source points cannot be isolated, so that for every $o^1 \in S^1$ there is some line O that connects it to some $o^2 \in S^2$. Thus any utility-weight pair in $\mathcal{V}(p^1)$ can be matched with some pair in $\mathcal{V}(p^2)$, or otherwise the representation of $\succ$ over $\mathcal{C}$ would conflict with that of its restriction to $\mathcal{L}(p^1)$.

Lemma 10 If $\succ$ satisfies Axioms 1-5, then for every $p^1, p^2 \in \mathcal{C}$, then $\phi^1 \in \Phi(p^1)$ if and only if there is $\phi^2 \in \Phi(p^2)$ such that $\pi\phi^1 \oplus (1-\pi)\phi^2 \in \Phi(\pi p^1 + (1-\pi)p^2)$ for every $\pi \in \mathbb{R}$.

Lemma 10 shows that given $p^1, p^2 \in \mathcal{C}$, for every $\phi^1 = (\alpha^1, \tau^1) \in \Phi(p^1)$ there exists some $q \in \mathcal{L}(p^2)$ such that $\phi^1 \in \Phi(\pi p^1 + (1-\pi)q)$ for every π, thus defining a source line parallel to the line between p^1 and q. Then defining $\phi^2 = (\alpha^2, \tau^2)$ such that $\frac{p^2 - \tau^2\zeta_{\alpha^2}}{1-\tau^2} = \frac{q - \tau^1\zeta_{\alpha^1}}{1-\tau^1}$, we have that $\pi\phi^1 \oplus (1-\pi)\phi^2 \in \Phi(\pi p^1 + (1-\pi)p^2)$ for every π. We can now extend this result to any finite number of dimensions by repeated application of this result. For $\phi^i = (\alpha^i, \tau^i)$ and $\pi^i \in \mathbb{R}$ for $i = 1, \ldots, n$, let the weighted linear combination be given by

$$\bigoplus_{i=1}^{n} \pi^i \phi^i = \pi^1 \phi^1 \oplus \pi^2 \phi^2 \oplus \ldots \oplus \pi^n \phi^n = \left(\frac{\sum_{i=1}^{n} \pi^i \tau^i \alpha^i}{\sum_{i=1}^{n} \pi^i \tau^i}, \sum_{i=1}^{n} \pi^i \tau^i \right)$$

The following lemma shows that given any collection of elements $\{p^1, \ldots, p^n\} \in \mathcal{C}$, there exists a collection of utility-weight pairs $\psi = (\phi^i, \ldots, \phi^n)$ that define a hyperplane on which every point is a source.

Lemma 11 If $\succ$ satisfies Axioms 1-5, then for every $\{p^1, \ldots, p^n\} \in \mathcal{C}$, there is $\psi = (\phi^1, \ldots, \phi^n) \in \mathbb{R}^n$ such that $\bigoplus_{i=1}^{n} \pi^i \phi^i \in \Phi(\sum_{i=1}^{n} \pi^i p^i)$ for every $(\pi^1, \ldots, \pi^n)$ such that $\sum_{i=1}^{n} \pi^i = 1$.

The result of Lemma 11 directly prescribes the construction of a multiple weighted expected utility representation for $\succ$ over $\mathcal{C}$. Let $\{p^1, \ldots, p^n\}$ denote a basis for $\mathcal{C}$ and let

$$\Psi = \left\{ (\phi^1, \ldots, \phi^n) : \bigoplus_{i=1}^{n} \pi^i \phi^i \in \Phi\left(\sum_{i=1}^{n} \pi^i p^i \right) \ \forall \ (\pi^1, \ldots, \pi^n) \in \mathbb{R}^n, \sum_{i=1}^{n} \pi^i = 1 \right\}$$

For every $\psi = ((\alpha^1, \tau^1), \ldots, (\alpha^n, \tau^n)) \in \Psi$, we have that there are source points projecting incomparability planes over $\mathcal{L}(p^i)$ located at $o^i = \frac{p^i - \tau^i \zeta_{\alpha^i}}{1 - \tau^i}$. Furthermore for every $q = \sum_{i=1}^{n} \pi^i p^i$, there is a source point for $\mathcal{L}(q)$ located at

$$o = \frac{\sum_{i=1}^{n} \pi^i p^i - [\sum_{i=1}^{n} \pi^i \tau^i] \zeta_{\frac{\sum_{i=1}^{n} \pi^i \tau^i \alpha^i}{\sum_{i=1}^{n} \pi^i \tau^i}}}{1 - \sum_{i=1}^{n} \pi^i \tau^i} = \frac{\sum_{i=1}^{n} \pi^i (p^i - \tau^i \zeta_{\alpha^i})}{\sum_{i=1}^{n} \pi^i (1 - \tau^i)} = \frac{\sum_{i=1}^{n} \pi^i (1 - \tau^i) o^i}{\sum_{i=1}^{n} \pi^i (1 - \tau^i)}$$

Thus we can construct the functions $U^\psi : \mathcal{C} \mapsto \mathbb{R}$ and $W^\psi : \mathcal{C} \mapsto \mathbb{R}$ by letting for every $q = \sum_{i=1}^{n} \pi^i p^i$,

$$U^\psi(q) = \frac{\sum_{i=1}^{n} \pi^i \tau^i \alpha^i}{\sum_{i=1}^{n} \pi^i \tau^i} \qquad W^\psi(q) = \sum_{i=1}^{n} \pi^i \tau^i$$

By construction $(U^\psi(q), W^\psi(q)) \in \Phi(q)$ for every $q \in \mathcal{C}$ and as in the two-dimensional case, we have $U^\psi(\overline{p}) = 1$, $U^\psi(\underline{p}) = 0$ and $W^\psi(\overline{p}) = W^\psi(\underline{p}) = 1$. Furthermore, it is easily verified that (U^ψ, W^ψ) is weighted linear over $\mathcal{C}$. Collect these functions in $\mathcal{V} = \{(U^\psi, W^\psi) : \psi \in \Psi\}$. We can now show that $\mathcal{V}$ represents $\succ$ over $\mathcal{C}$, using similar arguments as those employed in the two-dimensional model.

Lemma 12 If $\succ$ satisfies Axioms 1-5, then for every $p^1, p^2 \in \mathcal{C}$, $p^1 \asymp p^2$ if and only if $U(p^1) = U(p^2)$ for some $(U, W) \in \mathcal{V}$.

Lemma 13 If $\succ$ satisfies Axioms 1-5, then for every $p^1, p^2 \in \mathcal{C}$, $p^1 \succ p^2$ if and only if $U(p^1) > U(p^2)$ for every $(U, W) \in \mathcal{V}$.

The main representation theorem thus follows directly from Lemma 13.

Theorem 1 A preference relation $\succ$ over a convex subset $\mathcal{C}$ of a finite dimensional linear space $\mathcal{L}$ is bounded and satisfies Axioms 1-5 if and only if there is a set $\mathcal{V}$ of weighted linear pairs such that for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow U(p) > U(q) \quad \forall\ (U, W) \in \mathcal{V}$$

Proof of Theorem 1 The proof of necessity is self-evident. The proof of sufficiency follows by constructing $\mathcal{V}$ as prescribed and applying Lemma 13. ■

Corollary to Theorem 1 A preference relation $\succ$ over $\Delta(X)$ is bounded and satisfies Axioms 1-5 if and only if there is a set $\tilde{\mathcal{V}}$ of utility-weight pairs such that for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)} > \frac{\sum_{x \in X} q(x) w(x) u(x)}{\sum_{x \in X} q(x) w(x)} \quad \forall\ (u, w) \in \tilde{\mathcal{V}}$$

Proof of Corollary For every weighted linear pair (U, W) over $\Delta(X)$, let $u(x) =$

$U(\delta_x)$ and $w(x) = W(\delta_x)$ for every $x \in X$, so that we have for every $p \in \Delta(X)$ that

$$U(p) = \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} \qquad W(p) = \sum_{x \in X} p(x)w(x)$$

The result follows immediately by applying Theorem 1 and letting $\tilde{\mathcal{V}} = \{(u, w) : (U, W) \in \mathcal{V}\}$. ■

2.3 Conclusion

In this chapter, we have established that simultaneously relaxing completeness, as in Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13], and weakening independence to the ratio substitution property of Chew and MacCrimmon (1979) [7] yields preferences that have a representation by a set of weighted linear utility functions. Notably, we showed that although obtaining the multi-utility representation relied on applying the independence axiom, and the weighted utility representation on completeness, we can build a general model that admits the results of both without making any additional assumptions to impose structure.

Our model generates preferences that exhibit the characteristics of both the multi-utility and weighted utility models while generating phenomena that do not arise in either. In the two-dimensional triangle diagram of Machina (1982) [22], we have multiple source points, each projecting a set of incomparability curves, so that a lottery p is strictly preferred to an alternative q if and only if it lies above each incomparability curve intersecting q. Hence the overall indifference map is given by the superimposition of several weighted utility indifference maps. Extended to any mixture space of finite dimension, we have a set of source hyperplanes from which originate sets of incomparability planes that are not necessarily parallel, so that the shapes of the upper and lower contour sets of any mixture may vary throughout.

In the final chapter, we will provide some intuition to complement the technical results of this chapter. In particular, we will investigate the special cases where a decision maker's set of criteria are given either by a set of utility functions paired with a single weight functions, or a single utility function and multiple weight functions, and consider the patterns of behavior that arise in either instance. Notably, we show that although these cases may appear to be indistinguishable from each other, they have dramatically different consequences on the relationship between risk and indecision.

3. INCOMPLETE PREFERENCES WITH CONFLICTING TASTES AND PERCEPTIONS

3.1 Motivation

In the preceding chapter, we showed that preferences that are neither negatively transitive, and hence complete, nor satisfy independence, but only a weaker ratio substitution property, have a utility representation that exhibits both multiplicity and non-linearity, consisting of a set of weighted linear utility functions all of which must agree on a ranking of two alternatives in order for the decision maker to show a strict preference. That is, if $\Delta(X)$ is the set of probability distributions over a finite outcome set X representing all the lotteries over these prizes, we have that there is a set $\tilde{\mathcal{V}}$ consisting of pairs of utility $u : X \mapsto \mathbb{R}$ and weight $w : X \mapsto \mathbb{R}^+$ functions such that for every $p, q \in \Delta(X)$, we have

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)} \quad \forall\ (u, w) \in \tilde{\mathcal{V}}$$

This model thus combines the multi-utility models of Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13] with the weighted utility model of Chew and MacCrimmon (1979) [7] into a single setup that can admit all the conclusions of either. However, the range of phenomena encompassed by this framework is not restricted merely to behavior that can already be explained by existing models that relax either completeness or independence alone. This chapter will focus on some special cases of the multiple weighted utility model that illustrate the types of attitudes that decision makers whose preferences are given by such a representation may have, especially with regard to how the tastes and perceptions that the various utility functions exhibit will shift in relation to one another, and the consequences this has to the degree of incompleteness, a notion which has been repeatedly referenced in the preceding chapters but has yet to be fully explored in detail.

Each $(u, w) \in \tilde{\mathcal{V}}$ can be thought of as representing some set of priorities for the decision maker. In the lucky event that they all indicate a particular lottery p should be ranked higher than an alternative q, then she is able to decisively pick p in the knowledge that no aspect of her internal thought process would have rather chosen q instead. Whenever any of these disagree on a ranking, however, then she considers the alternatives to be incomparable, so picking either one or the other would leave her dissatisfied in some way, a circumstances which in all likelihood describes the vast majority of choices that people must make in reality, no matter what kind of qualities our idealized self-images might possess. Furthermore, each of the criteria evaluates lotteries as a weighted expected value, with the weight assigning a level of importance $w(x)$ to every outcome, separate from the utility value $u(x)$. The weight function can thus be interpreted as a distortion function that gives the decision maker a skewed perception of risk, transforming the objective probabilities $p \in \Delta(X)$ to some subjective perceived probabilities $p^w \in \Delta(X)$ where for $x \in X$ we have

$$p^w(x) = \frac{p(x)w(x)}{\sum_{\xi \in X} p(\xi)w(\xi)}$$

This transformation will shift the perception of p so that the more highly weighted outcomes will appear more likely to be received than they are, regardless of whether such prizes are actually deemed desirable or not. For example, the Allais paradox might be explained if the decision maker attaches greater weight to the extreme outcomes of \$5m and \$0 than to the median outcome of \$1m, so that whatever the utility function indicates about her actual tastes, her actual degree of risk aversion will vary. Since any mixture of \$1m and \$0 is evaluated as if it gives a non-zero payoff with lower probability than it actually does, the decision maker will appear more willing to take on additional risk than her utility function over outcomes would indicate, readily exchanging the chance of receiving \$1m for a lower probability of winning \$5m at a ratio that would be unacceptable if the \$1m payoff were received with certainty. Likewise, mixtures of \$1m and \$5m are perceived as yielding \$5m with higher probability than they actually do, making the decision maker appear more risk averse.

Under incompleteness, the decision maker may have multiple weight functions, each representing a distinct transformation of the space of lotteries, and therefore may be unsure of her perception of risk, much like in Maccheroni (2004) [20], except that here the utility function representing her tastes need not be linear in either the payoffs or the probabilities, nor unique for that matter so that in addition to incomplete perception she may also exhibit incomplete tastes as in Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni [13]. Moreover, within the space of decision criteria $\tilde{\mathcal{V}}$, there may be several utility functions paired with a single weight function, several weight functions paired with a single utility, or both simultaneously, potentially in a manner that such that the sets of utilities and weights cannot be separated from each other so that $\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \times \tilde{\mathcal{W}}$. Therefore, indecision may arise from conflicts in tastes, perceptions, or both, and as we will show, within this framework the decision criteria may fall in and out of agreement with one another when evaluating different lotteries, thus allowing the level of decisiveness to vary depending on the particular choice being presented.

3.2 Interpretation

To illustrate the possibilities, we consider once again constructing a triangle diagram after Machina (1982) [22]. For every lottery $p \in \mathcal{C}$, the set of utility-weight pairs is given by

$$\Phi(p) = \left\{ (\alpha, \tau) : \beta p + (1-\beta) r \asymp \frac{\beta \tau \zeta_\alpha + (1-\beta) r}{\beta \tau + (1-\beta)} \; \forall \, \beta \in (0,1), r \in \mathcal{C} \right\}$$

Thus $(\alpha, \tau) \in \Phi(p)$ if and only if $p \asymp \zeta_\alpha$ and $\tau \geq 0$ satisfies ratio substitution between them, so that a set of incomparability curves is projected from the source point $o = \frac{p - \tau \zeta_\alpha}{1-\tau}$. Further recall that the location of any source point can be determined by drawing a similar triangle with vertices p, $\overline{o} = \frac{p - \tau \overline{p}}{1-\tau}$, and $\underline{o} = \frac{p - \tau \underline{p}}{1-\tau}$, and letting $o = \alpha \overline{o} + (1-\alpha) \underline{o}$ so that the source lies in the same position on the opposite edge as $\zeta_\alpha = \alpha \overline{p} + (1-\alpha) \underline{p}$ does on the best-worst edge. Hence the utility-weight pair (α, τ) defines a coordinate system for the set of source points.

Figure 3.1 depicts the triangle diagram with incomparability curves projected from three source points $o^i = \frac{p - \tau^i \zeta_{\alpha^i}}{1-\tau^i}$ for $i = 1, 2, 3$. Each pair (α^i, τ^i) defines utility and weight functions (U^i, W^i) over $\mathcal{L}(p)$ such that for every $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$ we have that

$$U^i(q) = \frac{\lambda \tau^i \alpha^i + (1-\lambda)\theta}{\lambda \tau^i + (1-\lambda)} \qquad W^i(q) = \lambda \tau^i + (1-\lambda)$$

Since we have that $q = W^i(q) \zeta_{U^i(q)} + (1 - W^i(q)) o^i$ for every q, the utility is defined by drawing a line from o^i through q to intersect the best-worst line, and the weight given by the position of q along this line. First observe that since $\tau^1 = \tau^2$, the source points o^1 and o^2 lie on the same vertical line parallel to $\overline{p} - \underline{p}$ and therefore generate identical weight functions so that $W^1(q) = W^2(q)$ for every q. Since $\alpha^2 > \alpha^1$, we

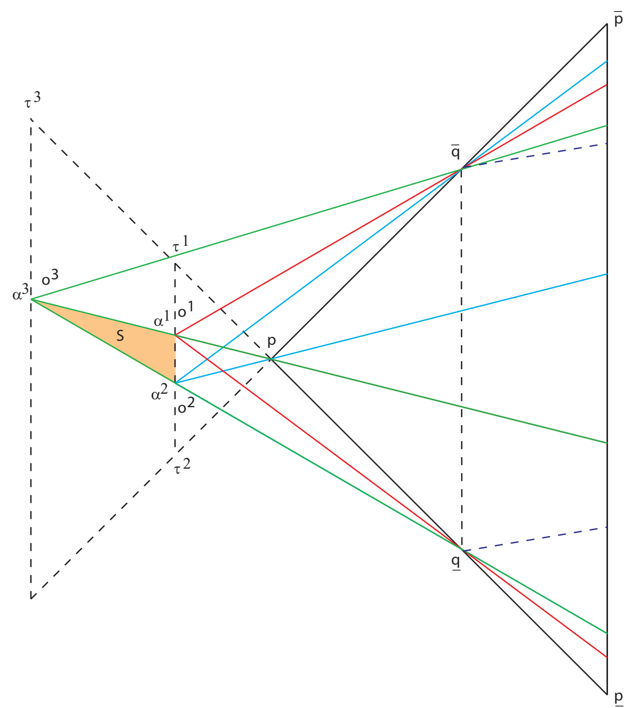


Fig. 3.1: Triangle Diagram with Three Source Points

have $U^2(q) > U^1(q)$ for every q and hence the incomparability curves originating from o^2 are steeper everywhere than those originating o^1, and consequently never coincide with one another. This implies that U^2 is comparatively more risk averse than U^1 and hence will always demand greater "compensation" for taking on additional risk in the sense of requiring a better chance of winning the best prize in order to be willing to gamble. Intuitively, two source points lying on the same vertical line indicate decision criteria that differ only in tastes and not perception, and therefore will never exactly agree on how to evaluate any particular lottery.

Now observe that $\alpha^1 = \alpha^3$, so that o^1 and o^3 both lie on the line connecting p to $\zeta_{\alpha^1} = \zeta_{\alpha^3}$ and therefore assign the same utility to this lottery $U^1(p) = U^3(p)$, but not necessarily to every other lottery in the simplex. Even though both criteria assign equal value to p, since $\tau^1 < \tau^3 < 1$ the incomparability curves originating from o^1 fan out more than those from o^3, reflecting that the former represent a more distorted weight scheme from the objective probabilities than the latter, thus deviating more from the independence axiom and exhibiting more extreme Allais-type behavior. As both share identical tastes however, they assign equal value to p itself and thus the incomparability curves through p originating from o^1 and o^3 exactly coincide, dividing the simplex in two such that U^1 is more risk averse than U^3 on the upper half and less risk averse on the lower half. Fixing some β and letting $\overline{q} = \beta p + (1-\beta)\overline{p}$ and $\underline{q} = \beta p + (1-\beta)\underline{p}$, we see that $U^1(\overline{q}) > U^3(\overline{q})$, but $U^1(\underline{q}) < U^3(\underline{q})$. Because $W^1(p) < W^3(p)$, U^1 sees $\overline{q}$ as being more similar to $\overline{p}$ than does U^3 and hence better, while simultaneously viewing $\underline{q}$ as closer to $\underline{p}$ and therefore worse.

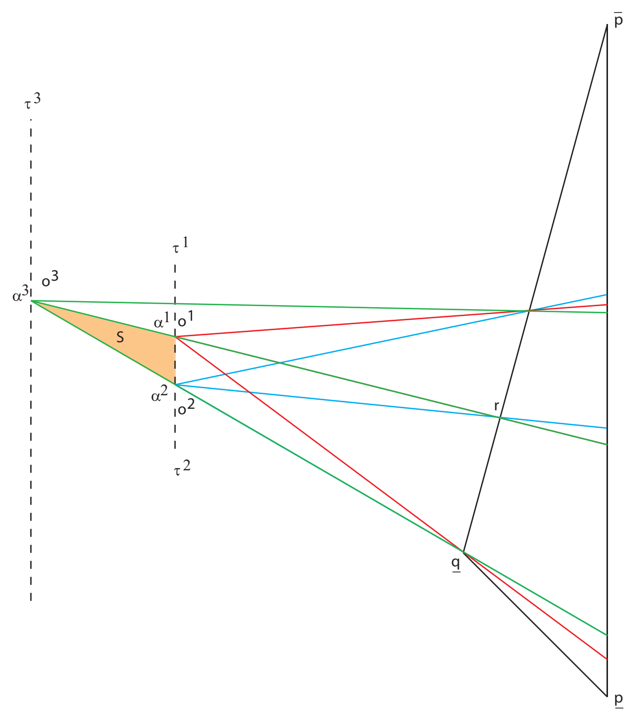


Fig. 3.2: Alternative Triangle Diagram with Three Source Points

Finally, consider the last possible pairing of source points o^2 and o^3, and note that since $\alpha^2 > \alpha^3$ but $\tau^2 < \tau^3$, these represent both different tastes and different perceptions. Observe however that this is not fundamentally different from the case where only perception varies, as we have $U^2(\underline{q}) = U^3(\underline{q})$ and thus the coinciding incomparability curves through $\underline{q}$ once again divide the simplex in two, with the lines originating from o^2 being steeper above and those from o^3 steeper below. Intuitively when evaluating $\underline{q}$, U^2 considers the component of p in this mixture to be more valuable than does U^3, but also assigns it less weight relative to the $\underline{p}$ component, so that for this particular lottery these competing effects exactly cancel each other out. As shown in Figure 3.2, we may define the simplex relative to $\underline{q}$ rather than p, and since $\mathcal{L}(\underline{q}) = \mathcal{L}(p)$ we can draw the same incomparability curves but with a different interpretation of how the various source points relate to one another. Here we have instead that o^2 and o^3 agree on taste but not perception of $\underline{q}$, while o^1 and o^3 disagree on both, though the incomparability curves they project still coincide at any $r = \gamma p + (1-\gamma)\zeta_{\alpha^1}$ and thus still divide the simplex in two as before. Note however that the relationship between o^1 and o^2 has not changed, as the weight functions W^1 and W^2 were identical everywhere and continue to be so here, so that U^1 and U^2 remain in conflict only in their relative attitude toward risk and not their perception of it.

This of course is quite an elementary result that simply draws on the definition of parallelism, as whenever the line between any two source points is not parallel to the line connecting $\overline{p}$ and $\underline{p}$, it can be extended to intersect it at some ζ_α, such that any lottery on this line is assigned the same utility α. Nevertheless, the case in which the set of source points on any given plane $\mathcal{L}(p)$ does consist only of a line parallel to $\overline{p} - \underline{p}$ appears to be worth investigating and, as we will show, represents manifestations of incompleteness arising only from differences in tastes and not perception.

3.3 *Incomplete Tastes Model*

In the preceding section, we saw that a pair of source points o^1, o^2 generate identical weight functions whenever $o^1 - o^2$ is parallel to $\overline{p} - \underline{p}$. A simple strengthening to the weak substitution axiom provides the necessary and sufficient condition for all pairs of source points to obey this property.

Axiom 6 (Parallel Substitution) For every $p, q \in \mathcal{C}$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ there is a unique $\gamma \in (0,1)$ such that for every $r \in \Delta(X)$, $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$.

Axiom 6 therefore ensures that given any two incomparable pairs $p \asymp q$, there exists a unique odds ratio $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$ that maintains this incomparability under mixture with a third alternative r. This effectively restricts the source space on any plane $\mathcal{L}(p)$ to consist only of a vertical line, so that incompleteness is restricted only to the utility levels.

Lemma 14 If $\succ$ satisfies Axioms 1-4,6 then for every $p \in \Delta(X)$, there is $\tau \geq 0$ such that $\Phi(p) = \{(\alpha, \tau) : \alpha \in A(p)\}$.

Now consider the restriction of $\succ$ to some plane $\mathcal{L}(p)$ for any $p \in \mathcal{C}$. Let $\tau(p)$ denote the unique weight, and recall that by Lemma 4 the set of utility values is given by a closed interval $A(p) = [\underline{\alpha}(p), \overline{\alpha}(p)]$, so that by Lemma 14, $\Phi(p) = [\underline{\alpha}(p), \overline{\alpha}(p)] \times \{\tau(p)\}$. A useful result here is that, just as in multi-utility with independence, the restriction of $\succ$ to $\mathcal{L}(p)$ has a parsimonious representation by the pair of utility functions indicating the highest and lowest levels of risk aversion, corresponding to $\overline{\alpha}(p)$ and $\underline{\alpha}(p)$, respectively. For every $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$, define

$$\overline{U}(q) = \frac{\lambda\tau(p)\overline{\alpha}(p) + (1-\lambda)\theta}{\lambda\tau(p) + (1-\lambda)} \qquad \underline{U}(q) = \frac{\lambda\tau(p)\underline{\alpha}(p) + (1-\lambda)\theta}{\lambda\tau(p) + (1-\lambda)}$$

$$W(q) = \lambda\tau(p) + (1-\lambda)$$

By construction, both $(\overline{U}, W)$ and $(\underline{U}, W)$ are elements of the full set of decision criteria $\mathcal{V}(p) = \{(U^\phi, W^\phi) : \phi \in \Phi(p)\}$, and it is easily verified that both are weighted linear as well. The following lemma shows that $\succ$ has a representation over $\mathcal{L}(p)$ simply by considering these two pairs rather than the entirety of $\mathcal{V}(p)$

Lemma 15 If $\succ$ satisfies Axioms 1-4,6 then for every $q^1, q^2 \in \mathcal{L}(p)$, $q^1 \succ q^2$ if and only if $\overline{U}(q^1) > \overline{U}(q^2)$ and $\underline{U}(q^1) > \underline{U}(q^2)$.

Now consider $p^1, p^2 \in \mathcal{C}$ such that $p^2 \notin \mathcal{L}(p^1)$. Recall that in the general model, we took great care to emphasize that source points across planes could not simply be connected arbitrarily, so that $\phi^i \in \Phi(p^i)$ for $i = 1, 2$ did not necessarily imply that we necessarily have $\pi\phi^1 \oplus (1-\pi)\phi^2 \in \Phi(\pi p^1 + (1-\pi)p^2)$. Under the assumption of parallel substitution however, we at least have the result that the unique weight $\tau(\cdot)$ is a linear function, implying that the space of source points lies on a hyperplane parallel to $\overline{p} - \underline{p}$, and that the upper and lower envelopes of the source space represented by $\overline{\alpha}(\cdot)$ and $\underline{\alpha}(\cdot)$ are weighted convex and concave, respectively.

Lemma 16 If $\succ$ satisfies Axioms 1-4,6, then for every $p^1, p^2 \in \mathcal{C}$ and $\pi \in [0, 1]$ we have that

$$\begin{aligned}
\tau(\pi p^1 + (1-\pi)p^2) &= \pi\tau(p^1) + (1-\pi)\tau(p^2) \\
\overline{\alpha}(\pi p^1 + (1-\pi)p^2) &\leq \frac{\pi\tau(p^1)\overline{\alpha}(p^1) + (1-\pi)\tau(p^2)\overline{\alpha}(p^2)}{\pi\tau(p^1) + (1-\pi)\tau(p^2)} \\
\underline{\alpha}(\pi p^1 + (1-\pi)p^2) &\geq \frac{\pi\tau(p^1)\underline{\alpha}(p^1) + (1-\pi)\tau(p^2)\underline{\alpha}(p^2)}{\pi\tau(p^1) + (1-\pi)\tau(p^2)}
\end{aligned}$$

As shown in Figure 3.3 shows, for $i = 1, 2$ the restriction of $\succ$ to each $\mathcal{L}(p^i)$ has a parsimonious representation by two utility weight pairs $(\overline{\alpha}^i, \tau^i)$ and $(\underline{\alpha}^i, \tau^i)$, as the incomparability curves projected by any other pair (α, τ^i) for a utility $\alpha \in (\underline{\alpha}^i, \overline{\alpha}^i)$ would always lie between the curves corresponding to the extreme values. As each lottery has only a single weight value, the source space for any $p^\pi = \pi p^1 + (1-\pi)p^2$ must be a vertical line, and as the weight function is linear, the entire source space must lie on some plane parallel to $\overline{p} - \underline{p}$. Furthermore, the utility range for each p^π must lie within the interval defined by the maximal and minimal permitted values $\overline{\alpha}^* = \frac{\pi\tau^1\overline{\alpha}^1 + (1-\pi)\tau^2\overline{\alpha}^2}{\pi\tau^1 + (1-\pi)\tau^2}$ and $\underline{\alpha}^* = \frac{\pi\tau^1\underline{\alpha}^1 + (1-\pi)\tau^2\underline{\alpha}^2}{\pi\tau^1 + (1-\pi)\tau^2}$, which ensures that upper and lower contour sets are convex as mixture dominance requires, and can be separated by the

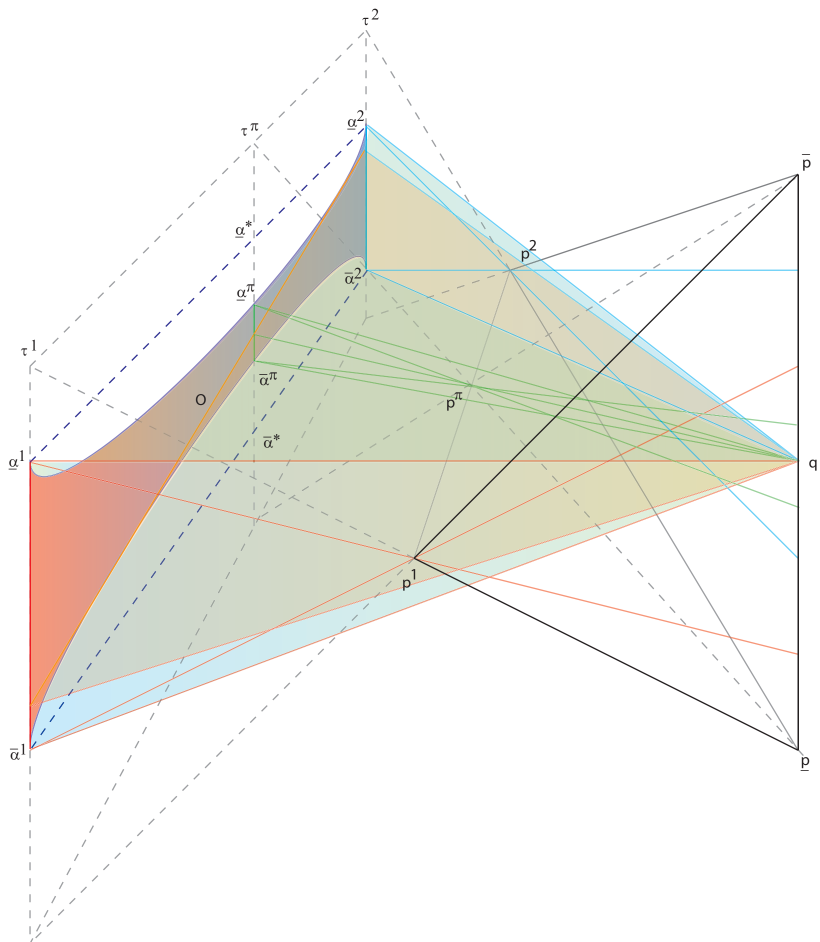


Fig. 3.3: Incomplete Tastes

incomparability plane projected by any valid source line O. Note that in general, the preference relation $\succ$ over $\mathcal{C}$ does not have a parsimonious representation by a pair of utilities, one of which is everywhere more risk averse than the other, unless all of the conditions in Lemma 16 are satisfied at equality. In this case, setting $\overline{U}(p) = \overline{\alpha}(p)$, $\underline{U}(p) = \underline{\alpha}(p)$, and $W(p) = \tau(p)$ would yield weighted linear pairs and the upper and lower contour sets would be wedge-shaped.

Figure 3.4 also shows how under parallel substitution, the degree of indecision that the decision maker exhibits must in some sense decrease under mixture. For every $p \in \mathcal{C}$, let $\mathcal{U}(p)$ denote the set of local utility functions, defined as the dual of the upper contour set $B(p)$. Given the result of Lemma 16 that the range of utility

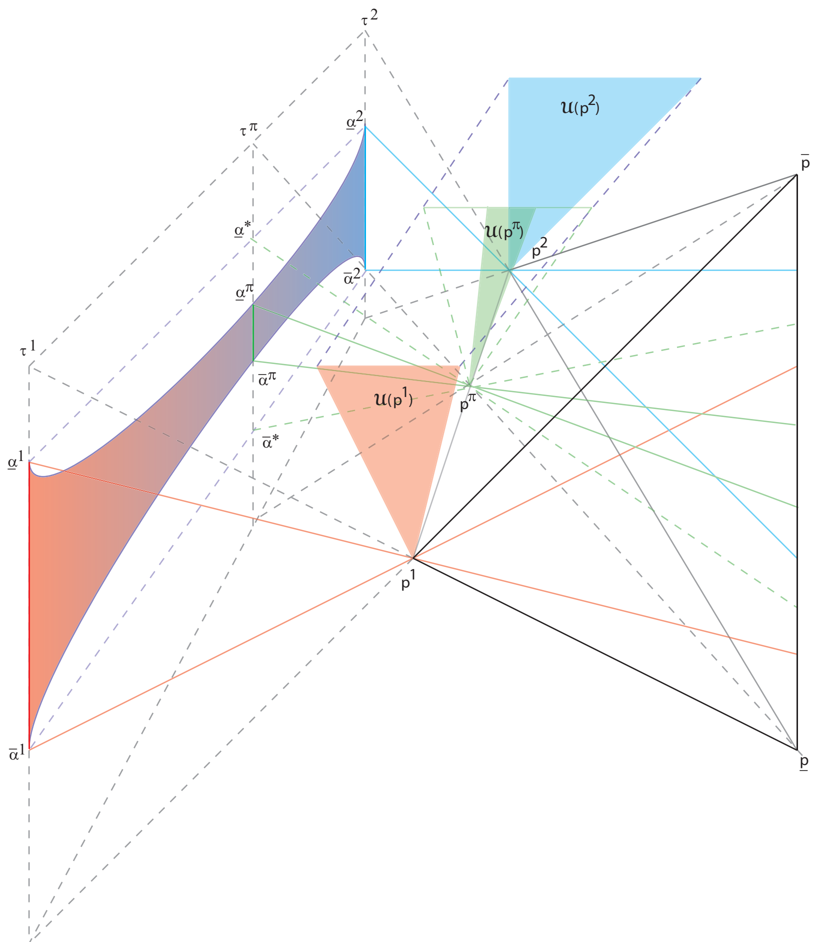


Fig. 3.4: Incompleteness Mitigated By Mixture

values p^π may take is restricted by the utility ranges for p^1 and p^2, we see that the decision maker can be no more unsure her evaluation of the mixture than she is of its components, and indeed in this example is considerably more capable of determining her affinity for p^π than of either p^1 or p^2. Intuitively, the maximal utility range is obtained whenever $\overline{\alpha}(p)$ and $\underline{\alpha}(p)$ are themselves weighted linear, which implies that the decision maker has only two criteria $(\overline{U}, W)$ and $(\underline{U}, W)$ that differ only in risk attitude, so that $\overline{U}$ uniformly assigns a higher value to every possible lottery than does $\underline{U}$, and do not conflict in their perception. Therefore, any utility value that the decision maker may attach to p^π can be no worse than the value obtained by taking the weighted expectation of the worst possible value for each its components, and no better than the weighted expectation of the best component values.

When the conditions of Lemma 16 are satisfied with strict inequality, the decision criteria vary not only in their attitude toward risk but also how they rank the components p^1 and p^2 relative to each other. This allows disagreements between to potentially cancel each other out when lotteries are mixed together, which we observe whenever the source lines corresponding to two distinct criteria intersect, exactly agreeing on a utility value for some p^π. Therefore, individuals may actually be decisive when evaluating mixtures than they are of the individual components. For example, a car buyer may be unsure of how much she is willing to pay for a red sports car and equally unsure of how to value the same car painted blue, as perhaps she has peculiar neuroses regarding how the color of her vehicle affects her social status. By instead considering a lottery that gives either color with equal probability, she can simply evaluate the car on its actual performance characteristics and eliminate all paint-induced indecision.

Now let $\{p^1, \ldots, p^n\}$ denote a basis for $\mathcal{C}$, and denote the set of source hyperplanes as

$$\Psi = \left\{ (\phi^1, \ldots, \phi^n) : \bigoplus_{i=1}^{n} \pi^i \phi^i \in \Phi\left(\sum_{i=1}^{n} \pi^i p^i\right) \ \forall\ (\pi^1, \ldots, \pi^n) \in \mathbb{R}^n, \sum_{i=1}^{n} \pi^i = 1 \right\}$$

Note that as the source lines may be infinitely extended by betweenness, the π^i may indeed take on negative values, which has relevance if the basis consists of elements in the interior of $\mathcal{C}$. Recall that by Theorem 1, we obtain a multiple weighted expected utility representation by defining for every $\psi = \{(\alpha^i, \tau^i)\}_{i=1}^n$ a utility and weight functions such that $U^\psi(p^i) = \alpha^i$ and $W^\psi(p^i) = \tau^i$ for $i = 1, \ldots, n$ and letting $\mathcal{V} = \{(U^\psi, W^\psi) : \psi \in \Psi\}$. Since further imposing the parallel substitution condition ensures that each lottery has a unique weight value, the representation theorem for the case where incompleteness is restricted only to tastes follows directly.

Theorem 2 A preference relation $\succ$ over a convex subset $\mathcal{C}$ of a finite dimensional linear space $\mathcal{L}$ is bounded and satisfies Axioms 1-4,6 if and only if there is a set $\mathcal{U}$ of utility functions and a weight function W such that (U, W) is a weighted linear pair for every $U \in \mathcal{U}$ and for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow U(p) > U(q) \quad \forall\, U \in \mathcal{U}$$

Proof of Theorem 2 Define $W : \mathcal{C} \mapsto \mathbb{R}^+$ such that $W(p) = \tau(p)$ for every $p \in \mathcal{C}$. Thus constructing $\mathcal{V}$ as prescribed we have by Lemma 14 that for every $(U^\psi, W^\psi) \in \mathcal{V}$, we have $W^\psi(p) = W(p)$ for every $p \in \mathcal{C}$. Letting $\mathcal{U} = \{U^\psi : (U^\psi, W) \in \mathcal{V}\}$ and applying Theorem 1 completes the proof. ■

Corollary to Theorem 2 A preference relation $\succ$ over $\Delta(X)$ is bounded and satisfies Axioms 1-4,6 if and only if there is a set $\tilde{\mathcal{U}}$ of utility functions and a weight function w such that for every $p, q \in \mathcal{C}$,

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)} \quad \forall\, u \in \tilde{\mathcal{U}}$$

Proof of Corollary Define the weight function $w : X \mapsto \mathbb{R}^+$ such that $w(x) = W(\delta_x)$ for every $x \in X$ and for every $U \in \mathcal{U}$ let $u(x) = U(\delta_x)$, then for every $p \in \Delta(X)$ we have

$$U(p) = \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} \qquad W(p) = \sum_{x \in X} p(x)w(x)$$

The result follows immediately by applying Theorem 2 and letting $\tilde{\mathcal{U}} = \{u : U \in \mathcal{U}\}$.
■

Observing the form of the representation in the corollary also gives us an impetus to establish a uniqueness result for this special case. Intuitively, a decision maker with incomplete tastes but only a single distortion function skewing her perception may alternatively be interpreted as simply applying a one-time transformation to the entire space of lotteries and subsequently adhering to the independence axiom with respect to the perceived probabilities, effectively ranking the transformed lotteries by a set of linear utilities, for which uniqueness results have been established in Dubra, Maccheroni, and Ok (2004) [10] and Galaabaatar and Karni (2012) [13]. For every $p = \sum_{x \in X} p(x)\delta_x \in \Delta(X)$, denote the transformed lottery $p^w \in \Delta(X)$ by

$$\begin{aligned} p^w(x) &= \frac{p(x)w(x)}{\sum_{\xi \in X} p(\xi)w(\xi)} \\ U(p) &= \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} = \sum_{x \in X} p^w(x)u(x) \end{aligned}$$

Define the transformed preference relation $\succ^w$ over $\mathcal{C}$ such that $p \succ q$ if and only if $p^w \succ^w q^w$, and the corresponding transformed weak preference $\succsim^w$, indifference $\sim^w$, and incomparability $\asymp^w$ relations using the given definitions taking $\succ^w$ as a primitive. By the corollary to Theorem 2, we have that

$$p^w \succ^w q^w \Leftrightarrow p \succ q \Leftrightarrow \sum_{x \in X} p^w(x)u(x) > \sum_{x \in X} q^w(x)u(x) \quad \forall\, u \in \tilde{\mathcal{U}}$$

Therefore, $\succ^w$ satisfies the independence axiom so that we are free to adapt established uniqueness results from multi-utility models here. Let $\langle \tilde{\mathcal{U}} \rangle$ denote the closure of the convex cone generated by the elements of $\tilde{\mathcal{U}}$ and a constant function on $\mathcal{L}$. The following theorem establishes the result that the set of utilities is unique to corresponding closed convex cone it generates.

Theorem 3 If a preference relation $\succ$ over $\Delta(X)$ satisfies Axioms 1-4,6 and there are sets of utilities $\tilde{\mathcal{U}}^i$ and weight functions $w^i : X \mapsto \mathbb{R}^+$ such that for $i = 1, 2$,

$$p \succ q \Leftrightarrow \frac{\sum_{x \in X} p(x)w^i(x)u^i(x)}{\sum_{x \in X} p(x)w^i(x)} > \frac{\sum_{x \in X} q(x)w^i(x)u^i(x)}{\sum_{x \in X} q(x)w^i(x)} \quad \forall\, u^i \in \tilde{\mathcal{U}}^i$$

Then $\langle \tilde{\mathcal{U}}^1 \rangle = \langle \tilde{\mathcal{U}}^2 \rangle$ and there is $k \in \mathbb{R}$ such that $w^2(x) = kw^1(x)$ for every $x \in X$.

Note that the uniqueness result only applies in the case of incomplete tastes, and not in the general case where perceptions may differ as well, as there is no easily defined analogue of the closed cone operation that can be applied to a set of decision criteria $\mathcal{V}$ that assigns multiple utility and weight values to each outcome. Suppose we have (U^1, W^1) and (U^2, W^2) and we wish to consider the analogue of the convex combination of these decision criteria. Consider defining for $\kappa \in [0,1]$ some weighted convex combination (U^κ, W^κ). In order to preserve the linearity of the weight function we must set

$$W^\kappa(p) = \kappa W^1(p) + (1-\kappa)W^2(p)$$

Then in order to maintain the weighted linearity of the utility function, we should have

$$U^\kappa(p) = \frac{\kappa W^1(p)U^1(p) + (1-\kappa)W^2(p)U^2(p)}{\kappa W^1(p) + (1-\kappa)W^2(p)}$$

However, a quick example shows that appending (U^κ, W^κ) to $\mathcal{V}$ may in fact alter the preferences. Suppose that for $p, q \in \mathcal{C}$, and $\kappa = \frac{1}{2}$, we have the values

	W^1	U^1	W^2	U^2	W^κ	U^κ
p	$\frac{1}{4}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{11}{24}$
q	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{13}{24}$

Thus $U^1(p) > U^1(q)$ and $U^2(p) > U^2(q)$, so that we should have $p \succ q$, but that $U^\kappa(p) < U^\kappa(q)$, so that the apparently harmless addition of (U^κ, W^κ) to the set of criteria would alter this strict preference to incomparability. As shown in Figure 3.5, this arises since the source lines O^1 and O^2 do not lie on the same plane, whereas in the special case of incomplete tastes they must necessarily be so. Note that as this is a representation of a three-dimensional diagram, the source lines do not actually intersect where they appear to do so and consequently the source space exhibits an

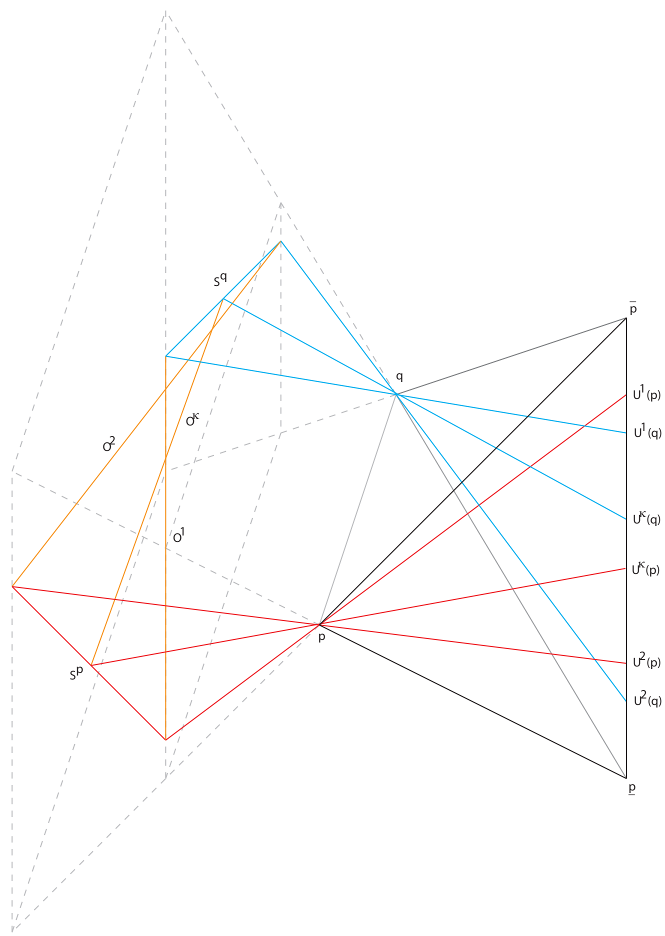


Fig. 3.5: Confounding of General Uniqueness Result

odd sort of twist as κ is varied, with the result that the weighted convex combination (U^κ, W^κ) may contradict its component decision criteria.

This curious situation could be considered a case of Simpson's paradox, where the success rate of a particular treatment may be higher than an alternative treatment within two sub-samples, but lower across the aggregated sample. In this context, the decision criteria represent the sub-samples, and the lotteries the treatments. For example suppose that a couple, with (U^1, W^1) representing the wife and (U^2, W^2) the husband, is deciding between buying a sports car p or a minivan q. Note that even though the wife would just as quickly as her husband abandon their children to satisfy a mid-life crisis, she assigns greater value to both choices while weighting the worse one more heavily, so that when the two attempt to reach a wholly unnecessary compromise, the resulting combination (U^κ, W^κ) would actually prescribe the opposite course of action from the one they had initially agreed upon. Despite this issue, we do expect that some form of uniqueness result does exist for the general model, and it is left as an open topic for future work.

3.4 *Incomplete Perception Model*

We now consider another special case, where incompleteness is restricted not to tastes but to perceptions, and is thus reflected by multiplicity in the weight, rather than the utility function. Recall that while the incomplete taste model was able to ensure that every lottery had a unique weight value, in this model we can only ensure a unique utility value along a line where the incomparability curves coincide to define an indifference curve. As shown before, when considering a general mixture space $\mathcal{C}$ any two source points that do not lie on a line parallel to $\overline{p} - \underline{p}$ define such an indifference curve, so we will restrict our attention to spaces of lotteries over prizes $\Delta(X)$. To obtain the desired representation, we impose the additional assumption that any degenerate outcome should be indifferent to some mixture of the best and worst outcomes.

Axiom 7 (Degenerate Solvability) For every $x \in X$, there is $\alpha \in [0, 1]$ such that $\delta_x \sim \zeta_\alpha$.

The effect of making this assumption is clear, since if $\delta_x \sim \zeta_\alpha$ then it must be that for any $\alpha' \neq \alpha$ we must have either $\delta_x \succ \zeta_{\alpha'}$ or $\delta_x \prec \zeta_{\alpha'}$. This implies that $A(\delta_x) = \{\alpha\}$ and therefore that $\Phi(\delta_x) = \{(\alpha, \tau) : \tau \in T(\delta_x, \zeta_\alpha)\}$, defining a set of source points that all lie on the extension of the line connecting δ_x to ζ_α. Note that unlike under incomplete tastes, this does not ensure that every lottery $p \in \Delta(X)$ has some mixture ζ_α to which it is indifferent, nor does it ensure that all of the source points lie on the same hyperplane.

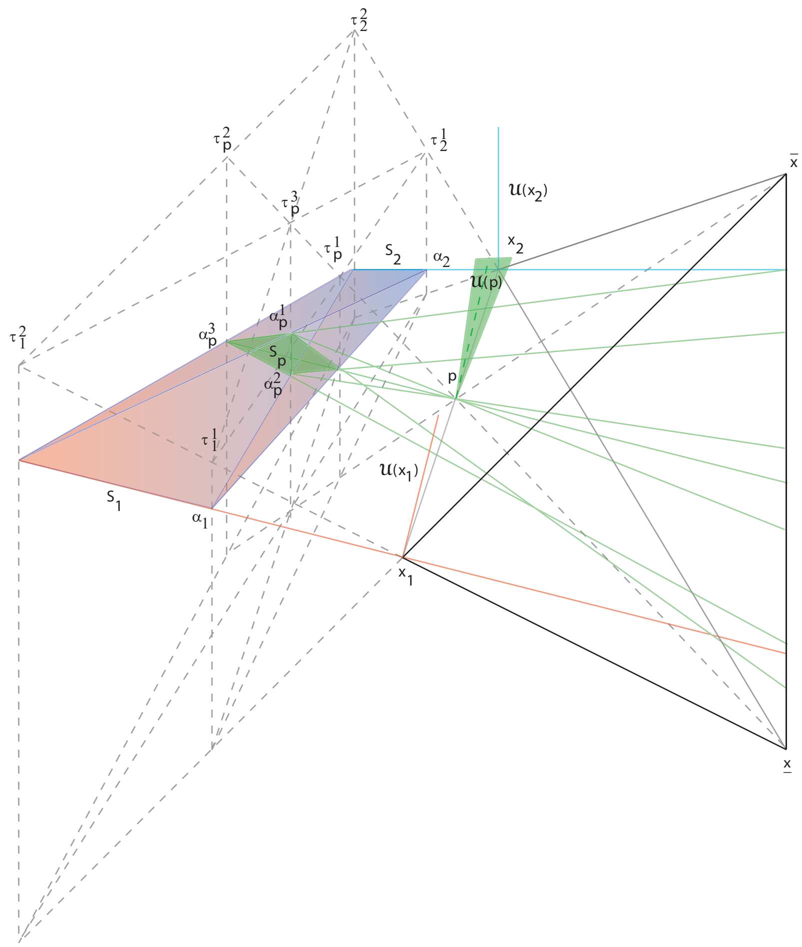


Fig. 3.6: Incomplete Perception

As shown in Figure 3.6, for $i = 1, 2$, we have $\delta_{x_i} \sim \zeta_{\alpha_i}$ so that these pairs of lotteries are connected by indifference curves, and that $T(\delta_{x_i}, \zeta_{\alpha_i}) = [\tau_i^1, \tau_i^2]$, so that there are a range of odds ratios satisfying substitution between them. Note however that since $\alpha_2 > \alpha_1$, these indifference curves do not lie on the same plane, and therefore for some $p = \pi\delta_{x_1} + (1 - \pi)\delta_{x_2}$, the four possible source lines connecting source points on $\mathcal{L}(\delta_{x_1})$ and $\mathcal{L}(\delta_{x_2})$ yield four distinct source points on $\mathcal{L}(p)$ at three distinct utility values and three separate weights. Note that any pair of such points, except the two corresponding to the same weight value τ_p^3, project incomparability curves that overlap somewhere on $\mathcal{L}(p)$. However, since the source points corresponding to the pairs (α_p^1, τ_p^3) and (α_p^2, τ_p^3) lie on a line parallel to the best-worst line, the incomparability curves they project never coincide as those originating from the latter are everywhere steeper than those originating from the former. Hence there are no indifference curves on $\mathcal{L}(p)$ so that no lottery lying on this plane has a unique utility value.

Consequently, with incomplete perception the degree of incompleteness is actually exacerbated by mixture, as there is a unique local utility function, given as before by the dual cone of the upper contour set, at both δ_{x_1} and δ_{x_2}, but a range of such functions at p. Note that the decision maker obtains the higher value α_p^2 for p under the scheme that weights the better outcome x_2 more heavily, thus perceiving p to be more similar to it than to x_1, the lower value α_p^1 that weights x_1 more, and a median value α_p^3 under any transformation that gives these prizes equal significance. This suggests that the the inability to rank lotteries originates not from a failure to evaluate the outcomes properly, but from not knowing which of the potential prizes to pay more attention to. For example, a decision maker may fully understand how much she likes either a full glass of water or an empty one, but be unsure whether to perceive an equal mixture of the two as being half-full or half-empty, yielding a range of values reflecting varying degrees of optimism or pessimism.

The representation theorem for the incomplete perception case follows by once again constructing Ψ and noting that if $\delta_x \sim \zeta_\alpha$, then for every $\psi \in \Psi$ the utility-weight pair (U^ψ, W^ψ) must set $U^\psi(\delta_x) = \alpha$. Note however that Axiom 7 can only ensure that there is a unique utility function over outcomes, if there was additionally a unique utility over lotteries, then preferences must be complete so that just as in Chew and MacCrimmon (1979) [7], the weight functions over both outcomes and lotteries would be unique as well.

Theorem 4 A preference relation $\succ$ over $\Delta(X)$ is bounded and satisfies Axioms 1-5,7 if and only if there is a utility function u and a set $\tilde{\mathcal{W}}$ of weight functions and such that for every $p, q \in \Delta(X)$,

$$p \succ q \Leftrightarrow \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)} \qquad \forall\, w \in \tilde{\mathcal{W}}$$

Proof of Theorem 4 Define $u : X \mapsto \mathbb{R}$ such that $\delta_x \sim \delta_{u(x)}$ for every $x \in X$, then constructing $\mathcal{V}$ as prescribed we have for every $(U^\psi, W^\psi) \in \mathcal{V}$ that $U^\psi(\delta_x) = u(x)$. Defining $w^\psi(x) = W^\psi(\delta_x)$ for every $\psi \in \Psi$ and letting $\tilde{\mathcal{W}} = \{w^\psi : (U^\psi, W^\psi) \in \mathcal{V}\}$ and applying Theorem 1 completes the proof. ■

Just as in the general model, a uniqueness proof for the incomplete perception case has so far proven elusive. If we were to construct some sort of convex cone generated by $\tilde{\mathcal{W}}$, then any combination within must itself be a linear function as the entire concept of weighted utility relies on the weight functions themselves being linear, though the value function is not itself linear in the weights. Recall that the uniqueness result in the incomplete tastes case was possible because Axiom 6 ensured that all of the source points lay on a single hyperplane, so that we could take convex combinations of the utility functions while staying on the same hyperplane. In order to have the same type of property here, all of the source points would need to lie on a single indifference hyperplane, which would in turn require that there exist some α such that $\delta_x \sim \zeta_\alpha$ for every $x \in X$, so that the utility function would be constant making the weight function entirely irrelevant. In the model of Maccheroni (2004) [20] that relaxed the completeness axiom within the dual theory of Yaari (1987) [25], the set of probability distortions was unique up to its closed and convex hull, though the utility was linear in payoffs in his case. It remains to be seen whether imposing some additional assumptions in this context will lead to a similar result.

3.5 Conclusion

We have considered special cases of the multiple weighted expected utility model where the conflict among the decision maker's various criteria are restricted to either tastes or perception alone. If the perception of risk is distorted only by a single weight function, then just as under independence, the restriction of preferences to a two-dimensional space $\mathcal{L}(p)$ has a parsimonious representation by a pair of utility functions reflecting the extremes of the possible risk attitudes, although here Allais-type behavior is still observed whenever perception strays from the objective reality. Over the entire space however, the utility functions represent not only the degree of risk aversion but also the relative ranking of the various outcomes, so that indecision is mitigated by mixture as it is entirely generated by an inability to rank outcomes. Thus the level of disagreement reaches its maximal extent when evaluating certain prizes and is no worse when considering lotteries instead. On the other hand, if the decision maker has fixed tastes but conflicting perceptions of risk, then indecision is exacerbated by mixture, as she can assign a single utility value to every component outcome of a lottery, but her inability to decide on a single transformation function results in disagreement on how to evaluate the lottery as a whole. In general, we should expect that multiplicity in both tastes and perception contribute to overall indecisiveness, such that the overall effect when considering mixtures is unclear and dependent on the exact structure of the representation.

Although the results obtained throughout the course of this analysis by no means represent a perfect model of decision making under risk, they represent a step toward building a broader understanding rather than considering each deviation from the norm as a separate problem requiring a specialized model incompatible with previous work. In the future, it may be worthwhile to establish a firmer uniqueness result, as well as consider relaxing the assumptions further, especially removing the structure imposed by the weak substitution axiom to obtain a general multiple non-expected utility model with a set of arbitrary decision criteria. Extending the framework to admit Knightian uncertainty in addition to measurable risk may also prove insightful, particularly because tastes will then be paired with subjective beliefs, which have a far more attractive interpretation than fundamentally incorrect perceptions of objective probabilities. Finally, we might wish to consider various decision rules that might be applied to resolve incompleteness whenever a decision maker is forced to choose between incomparable alternatives, and investigate the erratic patterns of behavior sure to result. Indeed, it would not be entirely erroneous to assert that human beings are never able to act without some degree of hesitation or indecisiveness, and that every choice we make would be considered the incorrect one by some facet of our personality that in another setting would be the deciding factor driving our actions. Framing effects are especially worthy of focus, as there is a degree of duality between lotteries and decision criteria, and if we are to apply weighting schemes to consider mixtures of the former, the same might well apply to the latter instead. While the inherent nature of incomplete preferences makes it quite a difficult subject to study that requires imposing structure on aspects of human behavior that are by definition not fully understood, the very ubiquity of hesitation and vacillation makes investigating the topic all the more critical.

Appendix A

PROOFS

Proof of Lemma 1 Fix $p, q, r \in \mathcal{C}$. For every $\alpha^* \in \{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$ we have by the Archimedean axiom that there are $\beta_1, \beta_2 \in [0,1]$ such that $(\alpha^* + \epsilon_1)p + (1 - \alpha^* - \epsilon_1)q \equiv \beta_1(\alpha^* p + (1-\alpha^*)q) + (1-\beta_1)p \succ r$ and $(\alpha^* - \epsilon_2)p + (1 - \alpha^* + \epsilon_2) \equiv \beta_2(\alpha^* p + (1-\alpha^*)q) + (1-\beta_2)q \succ r$. Picking some $\epsilon < \min\{\epsilon_1, \epsilon_2\}$ we have by mixture dominance that $N_\epsilon(\alpha^*) \subset (\alpha^* - \epsilon_2, \alpha^* + \epsilon_1) \subseteq \{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$. Hence $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$ is open and by a similar argument $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \prec r\}$ is open as well. Moreover $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \asymp r\} = [0,1] \setminus (\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\} \cup \{\alpha \in [0,1] : \alpha p + (1-\alpha)q \prec r\})$ is closed. ■

Proof of Lemma 2 Fix $p, q \in \mathcal{C}$ and $\alpha, \beta \in (0,1)$ such that $\alpha > \beta$ and let $\gamma = \frac{\beta}{\alpha}$. If $p \succ q$, then by two applications of betweenness we have that $p \succ \alpha p + (1-\alpha)q \succ \gamma(\alpha p + (1-\alpha)q) + (1-\gamma)q = \beta p + (1-\beta)q \succ q$. If $p \asymp q$, then by the above argument $\alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q$ implies $p \succ q$, and $\alpha p + (1-\alpha)q \prec \beta p + (1-\beta)q$ implies $p \prec q$, and hence we must have $\alpha p + (1-\alpha)q \asymp \beta p + (1-\beta)q$. ■

Proof of Lemma 3 Fix $p, q \in \mathcal{C}$ such that $p \asymp q$. Fix $r \in \mathcal{C}$ and pick $\beta, \gamma \in (0,1)$ that satisfy weak substitution so that $s \equiv \beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r \equiv t$. Now pick $\beta', \gamma' \in (0,1)$ such that $\tau \equiv \frac{\gamma'/(1-\gamma')}{\beta'/(1-\beta')} = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$, proving the proposition requires showing that $u \equiv \beta' p + (1-\beta)' r \asymp \gamma' q + (1-\gamma')r \equiv v$.

As depicted in Figure A.1, the extensions of the lines st and uv intersect at some

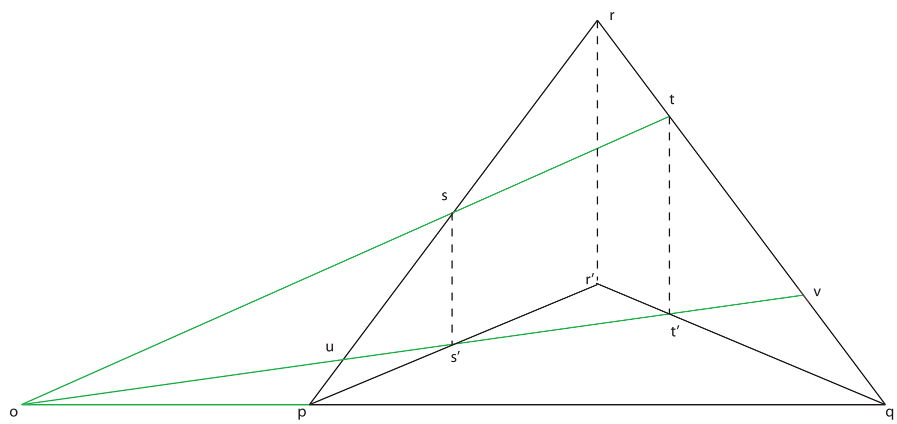


Fig. A.1: Applying Desargues' Theorem to Find the Source

source point lying outside of the simplex on the extended line pq, located at $o = \frac{p-\tau q}{1-\tau}$. Draw parallel lines from s and r such that the line from s intersects uv at some point s', and extending ps' intersects the line from r at some r', and let t' denote the intersection of uv and qr'. By Desargues' theorem, the triangles rst and $r's't'$ are perspective from the line opq, and hence the lines rr', ss', and tt' are parallel. This implies that $s' = \beta p + (1-\beta)r'$ and $t' = \gamma q + (1-\gamma)r'$, and hence by weak substitution that $s' \asymp t'$. Since both s' and t' lie on uv, we have by betweenness that $u \asymp v$ as well, completing the proof. ■

Proof of Lemma 4 Fix $p, q, r \in \mathcal{C}$ and define $\overline{\alpha} = \inf\{\alpha \in [0,1] : r \prec \alpha p + (1-\alpha)q\}$ and $\underline{\alpha} = \sup\{\alpha \in [0,1] : r \succ \alpha p + (1-\alpha)q\}$. By Lemma 2, $\alpha > \overline{\alpha}$ implies $p \prec \alpha p + (1-\alpha)q$ and $\alpha < \underline{\alpha}$ implies $p \succ \alpha p + (1-\alpha)q$, so that Axiom 1 implies that $\underline{\alpha} \leq \overline{\alpha}$, and furthermore that $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ implies $\neg(p \asymp \zeta_\alpha)$. By Lemma 1, $\{\alpha \in [0,1] : r \prec \alpha p + (1-\alpha)q\}$ and $\{\alpha \in [0,1] : r \succ \alpha p + (1-\alpha)q\}$ are open, so that $\alpha \geq \underline{\alpha}$ implies $\neg(r \succ \alpha p + (1-\alpha)q)$ and $\alpha \leq \overline{\alpha}$ implies $\neg(r \prec \alpha p + (1-\alpha)q)$. Hence $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ implies $r \asymp \alpha p + (1-\alpha)q$. ■

Proof of Lemma 5 Fix $p \in \mathcal{C}$ and pick any $(\alpha, \tau) \in \Phi(p)$. For every $q = \lambda p + (1 -$

$\lambda)\zeta_\theta \in \mathcal{L}(p)$, we have by ratio substitution that for every $\beta \in (0,1)$ and $r \in \mathcal{C}$,

$$\begin{aligned}
\beta q + (1-\beta)r = \beta\lambda p + \beta(1-\lambda)\zeta_\theta + (1-\beta)r &\asymp \frac{\beta\lambda\tau\zeta_\alpha + \beta(1-\lambda)\zeta_\theta + (1-\beta)r}{\beta\lambda\tau + \beta(1-\lambda) + (1-\beta)} \\
&\asymp \frac{\beta(\lambda\tau + (1-\lambda))\zeta_{\frac{\lambda\tau\alpha+(1-\lambda)\theta}{\lambda\tau+(1-\lambda)}} + (1-\beta)r}{\beta(\lambda\tau + (1-\lambda)) + (1-\beta)} \\
&\equiv \frac{\beta\tau'\zeta_{\alpha'} + (1-\beta)r}{\beta\tau' + (1-\beta)}
\end{aligned}$$

Hence by definition $(\alpha', \tau') \equiv \left(\frac{\lambda\tau\alpha+(1-\lambda)\theta}{\lambda\tau+(1-\lambda)}, \lambda\tau + (1-\lambda)\right) \in \Phi(q)$ and furthermore

$$\frac{p - \tau\zeta_\alpha}{1-\tau} = \frac{\lambda p + (1-\lambda)\zeta_\theta - \lambda\tau\zeta_\alpha - (1-\lambda)\zeta_\theta}{\lambda(1-\tau)} = \frac{q - \tau'\zeta_{\alpha'}}{1-\tau'}$$

To show the converse, apply the above letting $p = \frac{1}{\lambda}q + \left(1 - \frac{1}{\lambda}\right)\zeta_\theta$. ■

Proof of Lemma 6 Fix $q^1, q^2 \in \mathcal{L}(p)$. First suppose that $q^1 \asymp q^2$ and let $q^2 = \lambda q^1 + (1-\lambda)\zeta_\theta$. By betweenness, this implies that $q^1 \asymp \zeta_\theta$ and thus for some $\tau \geq 0$ we have $(\theta, \tau) \in \Phi(q^1)$. Therefore by Lemma 5, there is $(U, W) \in \mathcal{V}(p)$ such that $U(q^1) = \theta$. Since $U(\zeta_\theta) = \theta$, and (U, W) weighted linear by construction, this implies that $U(q^2) = U(q^1)$. Now suppose that there is $(U, W) \in \mathcal{V}(p)$ such that $U(q^1) = U(q^2) = \alpha'$ for some $\alpha' \in (0,1)$. Let $W(q^i) = \tau^i$ for $i = 1, 2$, then by construction we have that there is $\phi = (\alpha, \tau) \in \Phi(p)$ such that

$$\begin{aligned}
\frac{p - \tau\zeta_\alpha}{1-\tau} = \frac{q^1 - \tau^1\zeta_{\alpha'}}{1-\tau^1} &= \frac{q^2 - \tau^2\zeta_{\alpha'}}{1-\tau^2} \\
q^2 = \frac{1-\tau^2}{1-\tau^1}q^1 &+ \frac{\tau^2 - \tau^1}{1-\tau^1}\zeta_{\alpha'}
\end{aligned}$$

Since $(\alpha', \tau^i) \in \Phi(q^i)$ for $i = 1, 2$, $q^1 \asymp \zeta_{\alpha'}$ and thus by betweenness, $q^1 \asymp q^2$. ■

Proof of Lemma 7 Fix $q^1, q^2 \in \mathcal{L}(p)$. If $q^1 \succ q^2$ then by Lemma 6 there is no $(U, W) \in \mathcal{V}(p)$ such that $U(q^1) = U(q^2)$. Suppose that for some $(U, W) \in \mathcal{V}(p)$ we have that $U(q^1) < U(q^2)$, then letting $\tau^1 = W(q^1)$ there is $\beta \in (0,1)$ such that

$$U(q^2) = \frac{\beta W(q^1)U(q^1) + (1-\beta)W(\overline{p})U(\overline{p})}{\beta W(q^1) + (1-\beta)W(\overline{p})} = U(\beta q^1 + (1-\beta)\overline{p})$$

This implies that $q^2 \asymp \beta q^1 + (1-\beta)\overline{p}$ by Lemma 6, which contradicts mixture dominance. Hence $U(q^1) > U(q^2)$ for every $(U, W) \in \mathcal{V}(p)$. Now suppose that $U(q^1) > U(q^2)$ for every $(U, W) \in \mathcal{V}(p)$, this implies $\neg(q^1 \asymp q^2)$ by Lemma 6 and $\neg(q^1 \prec q^2)$ by the previous argument. Therefore, we must have that $q^1 \succ q^2$. ■

Proof of Lemma 8 Fix $p^1, p^2 \in \mathcal{C}$ and let $p^\pi = \pi p^1 + (1-\pi)p^2$ for $\pi \in \mathbb{R}$. Define $A^* = \bigcap_{\pi \in \mathbb{R}} A(p^\pi)$ such that $\alpha \in A^*$ if and only if $p^\pi \asymp \zeta_\alpha$ for every $\pi \in \mathbb{R}$. Suppose $p^1 \asymp p^2$, then if $A^* = \varnothing$ there are $\pi^1, \pi^2 \in \mathbb{R}$ and $\alpha' \in [0,1]$ such that $\underline{\alpha}(p^{\pi^1}) > \alpha' > \overline{\alpha}(p^{\pi^2})$ and hence $p^{\pi^1} \succ \zeta_{\alpha'} \succ p^{\pi^2}$. By mixture monotonicity, $\pi^1 > \pi^2$ implies $p^1 \succ p^2$, and $\pi^1 < \pi^2$ implies $p^1 \prec p^2$, neither of which is permissible, and hence $A^* \neq \varnothing$. Now suppose $\neg(p^1 \asymp p^2)$ and assume without loss of generality that $p^1 \succ p^2$. For every $\alpha \in [0,1]$, we have that by mixture monotonicity that $p^\pi \succ \zeta_\alpha$ for π sufficiently large, and hence $\alpha \notin A(p^\pi) \subseteq A^*$, which implies $A^* = \varnothing$. Hence $p \asymp q$ if and only if $A^* \neq \varnothing$. ■

Proof of Lemma 9 Fix $p^1, p^2 \in \mathcal{C}$ and let $p^\pi = \pi p^1 + (1-\pi)p^2$ for $\pi \in \mathbb{R}$. Define $\Phi^* = \bigcap_{\pi \in \mathbb{R}} \Phi(\pi p^1 + (1-\pi)p^2)$. First suppose that $\Phi = \varnothing$, then there are $\pi^1, \pi^2 \in \mathbb{R}$ such that $\Phi(p^{\pi^1}) \cap \Phi(p^{\pi^2}) = \varnothing$. This implies that for some $\delta \in \mathbb{R}$ we have without loss of generality that there is $\alpha' \in [0,1]$ and $\tau' \geq 0$ such that $\underline{\alpha}(p^{\pi^1} + \delta(\overline{p} - \underline{p})) > \alpha' + \frac{\delta}{\tau'} > \overline{\alpha}(p^{\pi^2} + \delta(\overline{p} - \underline{p}))$. This implies that $p^{\pi^1} + \delta(\overline{p} - \underline{p}) \succ \zeta_{\alpha' + \frac{\delta}{\tau'}} \succ p^{\pi^2} + \delta(\overline{p} - \underline{p})$, so that $\pi^1 > \pi^2$ implies $p^1 + \delta(\overline{p} - \underline{p}) \succ p^2 + \delta(\overline{p} - \underline{p})$ and $\pi^1 < \pi^2$ implies $p^1 + \delta(\overline{p} - \underline{p}) \prec p^2 + \delta(\overline{p} - \underline{p})$.

Now suppose that $\Phi^* \neq \varnothing$, then for every $(\alpha, \tau) \in \Phi^*$ we have by Lemma 5 that for every $\delta \in \mathbb{R}$, $p^\pi + \delta(\overline{p} - \underline{p}) \asymp \alpha + \frac{\delta}{\tau}$ for every $\pi \in \mathbb{R}$. This implies by Lemma 8 that $p^1 + \delta(\overline{p} - \underline{p}) \asymp p^2 + \delta(\overline{p} - \underline{p})$ for every $\delta \in \mathbb{R}$. We conclude that $\Phi \neq \varnothing$ if and only if

$p^1 + \delta(\overline{p} - \underline{p}) \asymp p^2 + \delta(\overline{p} - \underline{p})$ for every $\delta \in \mathbb{R}$. ■

Proof of Lemma 10 Fix $p^1, p^2 \in \mathcal{C}$ and pick any $\phi^1 = (\alpha^1, \tau^1) \in \Phi(p^1)$. Let $Q = \{q \in \mathcal{L}(p^2) : p^1 + \delta(\overline{p} - \underline{p}) \asymp q + \delta(\overline{p} - \underline{p})\ \forall\ \delta \in \mathbb{R}\}$ and let $q^\pi = \pi p^1 + (1-\pi)q$ for every $\pi \in \mathbb{R}$. For every $q \in Q$, we have by Lemma 9 that $\Phi^*_q = \bigcap_{\pi \in \mathbb{R}} \Phi(q^\pi) \neq \varnothing$. Suppose that there is no $q \in Q$ such that $\phi^1 \in \Phi^*_q$, then there is $\pi, \delta \in \mathbb{R}$ such that either $q^\pi + \delta(\overline{p} - \underline{p}) \succ \zeta_{\alpha^1 + \frac{\delta}{\tau^1}}$ for every $q \in Q$ or $q^\pi + \delta(\overline{p} - \underline{p}) \prec \zeta_{\alpha^1 + \frac{\delta}{\tau^1}}$ for every $q \in Q$. If the former, then for some $\epsilon > 0$ and $q \in Q$ we have $p^1 + \delta(\overline{p} - \underline{p}) \succ q^\pi + (\delta - \epsilon)(\overline{p} - \underline{p}) \succ \zeta_{\alpha^1 + \frac{\delta}{\tau^1}}$, and if the latter for some $\epsilon > 0$ and $q \in Q$ we have $p^1 + \delta(\overline{p} - \underline{p}) \prec q^\pi + (\delta + \epsilon)(\overline{p} - \underline{p}) \prec \zeta_{\alpha^1 + \frac{\delta}{\tau^1}}$, either of which contradict $(\alpha^1, \tau^1) \in \Phi(p^1)$. Thus there is $q \in Q$ such that $(\alpha^1, \tau^1) \in \Phi(q^\pi)$ for every $\pi \in \mathbb{R}$.

Since $q \in \mathcal{L}(p^2)$ there are $\lambda, \theta \in \mathbb{R}$ such that $p^2 = \lambda q + (1-\lambda)\zeta_\theta$. Let $\phi^2 = (\alpha^2, \tau^2) = \left(\frac{\lambda\tau^1\alpha^1 + (1-\lambda)\theta}{\lambda\tau^1 + (1-\lambda)}, \lambda\tau^1 + (1-\lambda)\right)$. Then for every $\pi \in \mathbb{R}$ we have that

$$\begin{aligned}
\pi p^1 + (1-\pi)p^2 &= \pi p^1 + (1-\pi)\lambda q + (1-\pi)(1-\lambda)\zeta_\theta \\
&= [\pi + (1-\pi)\lambda]\left[\frac{\pi p^1 + (1-\pi)\lambda q}{\pi + (1-\pi)\lambda)}\right] + (1-\pi)(1-\lambda)\zeta_\theta
\end{aligned}$$

For every $\pi \in \mathbb{R}$, $\phi^1 \in \Phi\left(\frac{\pi p^1 + (1-\pi)\lambda q^*}{\pi + (1-\pi)\lambda}\right)$ by construction so that by Lemma 5 we have that $\phi^\pi = (\alpha^\pi, \tau^\pi) \in \Phi(\pi p^1 + (1-\pi)p^2)$ where

$$\begin{aligned}
\alpha^\pi &= \frac{\lambda^\pi\tau^1\alpha^1 + (1-\lambda^\pi)\theta}{\lambda^\pi\tau^1 + (1-\lambda^\pi)} = \frac{\pi\tau^1\alpha^1 + (1-\pi)\lambda\tau^1\alpha^1 + (1-\pi)(1-\lambda)\theta}{\pi\tau^1 + (1-\pi)\lambda\tau^1 + (1-\pi)(1-\lambda)} \\
&= \frac{\pi\tau^1\alpha^1 + (1-\pi)[\lambda\tau^1 + (1-\lambda)]\left[\frac{\lambda\tau^1\alpha^1 + (1-\lambda)\theta}{\lambda\tau^1 + (1-\lambda)}\right]}{\pi\tau^1 + (1-\pi)[\lambda\tau^1 + (1-\lambda)]} = \frac{\pi\tau^1\alpha^1 + (1-\pi)\tau^2\alpha^2}{\pi\tau^1 + (1-\pi)\tau^2} \\
\tau^\pi &= \lambda^\pi\tau^1 + (1-\lambda^\pi) = \pi\tau^1 + (1-\pi)(\lambda\tau^1 + (1-\lambda)) = \pi\tau^1 + (1-\pi)\tau^2
\end{aligned}$$

Observing that $\phi^\pi = \pi\phi^1 \oplus (1-\pi)\phi^2$ for every $\pi \in \mathbb{R}$ completes the proof. ■

Proof of Lemma 11 Fix $\{p^1, \ldots, p^n\} \in \mathcal{C}$ and for $m \leq n$ define

$$\Psi(p^1, \ldots, p^m) = \left\{ (\phi^1, \ldots, \phi^m) : \bigoplus_{i=1}^{m} \pi^i \phi^i \in \Phi\left(\sum_{i=1}^{m} \pi^i p^i \right) \; \forall \; (\pi^1, \ldots, \pi^m) \in \mathbb{R}^m, \sum_{i=1}^{m} \pi^i = 1 \right\}$$

We will show by induction that $\Psi(p^1, \ldots, p^m) \neq \varnothing$ for every $m \leq n$. For the base case, note that $\Psi(p^1) = \Phi(p^1) \neq \varnothing$. Now suppose that $\Psi(p^1, \ldots, p^{m-1}) \neq \varnothing$. Let $Q = \{\sum_{i=1}^{m-1} \pi^i p^i : \sum_{i=1}^{m-1} \pi^i = 1\}$. Pick any $(\phi^1, \ldots, \phi^{m-1}) \in \Psi(p^1, \ldots, p^{m-1})$, then for every $q = \sum_{i=1}^{m-1} \pi^i p^i \in Q$, we have that $\phi^q = \bigoplus_{i=1}^{m-1} \pi^i \phi^i \in \Phi(q)$. By Lemma 10, this implies that there is $\phi^m \in \Phi(p^m)$ such that $\pi^* \phi^q \oplus (1-\pi^*) \phi^m \in \Phi(\pi^* q + (1-\pi^*) p^m)$ for every $\pi^* \in \mathbb{R}$. Thus, for every $(\pi^1, \ldots, \pi^m) \in \mathbb{R}^m$, pick $q = \frac{\sum_{i=1}^{m-1} \pi^i p^i}{\sum_{i=1}^{m-1} \pi^i} \in Q$ and $\pi^* = \frac{\sum_{i=1}^{m-1} \pi^i}{\sum_{i=1}^{m} \pi^i}$. The result above implies that $\bigoplus_{i=1}^{m} \pi^i \phi^i = \pi^* \phi^q \oplus (1-\pi^*) \phi^m \in \Phi(\pi^* q + (1-\pi)^* p^m) = \Phi(\sum_{i=1}^{m} \pi^i p^i)$, and hence $(\phi^1, \ldots, \phi^m) \in \Psi(p^1, \ldots, p^m)$. Therefore $\Psi(p^1, \ldots, p^m) \neq \varnothing$ for every $m \leq n$, and picking any $\psi \in \Psi(p^1, \ldots, p^n)$ completes the proof. ∎

Proof of Lemma 12 Fix $p^1, p^2 \in \mathcal{C}$. First suppose that $p^1 \asymp p^2$, then by Lemma 8 there is $\alpha \in [0, 1]$ such that $\pi p^1 + (1 - \pi) p^2 \asymp \zeta_\alpha$ for every $\pi \in \mathbb{R}$. Thus by Lemma 10 there are $\tau^1, \tau^2 \geq 0$ such that $(\alpha, \pi \tau^1 + (1 - \pi) \tau^2) \in \Phi(\pi p^1 + (1 - \pi) p^2)$ for every $\pi \in \mathbb{R}$, and hence for some $\psi \in \Psi$ we have $U^\psi(p^1) = U^\psi(p^2) = \alpha$. Now suppose that for some $(U, W) \in \mathcal{V}$ we have $U(p^1) = U(p^2)$ and hence $U(\pi p^1 + (1 - \pi) p^2) = \alpha$ for every $\pi \in \mathbb{R}$. This implies that $\pi p^1 + (1 - \pi) p^2 \asymp \zeta_\alpha$ for every $\pi \in \mathbb{R}$ and hence $p^1 \asymp p^2$ by Lemma 8. ∎

Proof of Lemma 13 Fix $p^1, p^2 \in \mathcal{C}$. First suppose that $p^1 \succ p^2$, then by Lemma 12 there is no $(U, W) \in \mathcal{V}$ such that $U(p^1) = U(p^2)$. Suppose that for some $(U, W) \in \mathcal{V}$ we have $U(p^1) < U(p^2)$, then letting $\tau^1 = W(p^1)$ there is $\beta \in (0, 1)$ such that

$$U(p^2) = \frac{\beta W(p^1) U(p^1) + (1 - \beta) W(\overline{p}) U(\overline{p})}{\beta W(p^1) + (1 - \beta) W(\overline{p})} = U(\beta p^1 + (1 - \beta) \overline{p})$$

Hence by Lemma 12, $p^2 \asymp \beta p^1 + (1-\beta)\overline{p}$, contradicting mixture dominance. Thus we must have $U(p^1) > U(p^2)$ for every $(U, W) \in \mathcal{V}$. Now suppose that $U(p^1) > U(p^2)$ for every $(U, W) \in \mathcal{V}$, this implies that $\neg(p^1 \asymp p^2)$ by Lemma 12 and $\neg(p^1 \prec p^2)$ by the argument above, and hence $p^1 \succ p^2$. ■

Proof of Lemma 14 Fix $p \in \mathcal{C}$ and for any $\epsilon \in \mathbb{R}$ let $p^\epsilon = p + \epsilon(\delta_{\overline{x}} - \delta_{\underline{x}})$ for every $\epsilon \in \mathbb{R}$. By Lemma 5 we have that $(\alpha, \tau) \in \Phi(p)$ if and only if $(\alpha + \frac{\epsilon}{\tau}, \tau) \in \Phi(p^\epsilon)$. Suppose that there are $(\alpha^1, \tau^1), (\alpha^2, \tau^2) \in \Phi(p)$ such that $\tau^1 \neq \tau^2$, then letting $\epsilon^* = \frac{\alpha^1 - \alpha^2}{\frac{1}{\tau^2} - \frac{1}{\tau^1}}$ we have that $\alpha^* \equiv \alpha^1 + \frac{\epsilon^*}{\tau^1} = \alpha^2 + \frac{\epsilon^*}{\tau^2}$. But this implies that $(\alpha^*, \tau^i) \in \Phi(p^{\epsilon^*})$ for $i = 1, 2$ so that for every $\beta \in (0, 1)$ we have

$$\beta p^{\epsilon^*} + (1-\beta) r \asymp \frac{\beta \tau^i \zeta_{\alpha^*} + (1-\beta) r}{\beta \tau^i + (1-\beta)} \equiv \gamma^i \zeta_{\alpha^*} + (1-\gamma^i) r \qquad \forall\, r \in \mathcal{C}$$

But $\tau^1 \neq \tau^2$ implies $\gamma^1 \neq \gamma^2$, contradicting Axiom 6. This implies that $\tau^1 = \tau^2$ for every $(\alpha^1, \tau^1), (\alpha^2, \tau^2) \in \Phi(p)$. ■

Proof of Lemma 15 Fix $q^1, q^2 \in \mathcal{L}(p)$. Suppose $q^1 \succ q^2$, then by Lemma 7, $U(q^1) > U(q^2)$ for every $(U, W) \in \mathcal{V}(p)$, so since $(\overline{U}, W), (\underline{U}, W) \in \mathcal{V}(p)$, $\overline{U}(q^1) > \overline{U}(q^2)$ and $\underline{U}(q^1) > \underline{U}(q^2)$. Now suppose that $\overline{U}(q^1) > \overline{U}(q^2)$ and $\underline{U}(q^1) > \underline{U}(q^2)$. Then for every $(U^\phi, W^\phi) \in \mathcal{V}(p)$, there is by construction $\phi = (\alpha', \tau') \in \Phi(p)$ such that for $q = \lambda p + (1-\lambda)\zeta_\theta \in \mathcal{L}(p)$,

$$U^\phi(q) = \frac{\lambda \tau' \alpha' + (1-\lambda)\theta}{\lambda \tau' + (1-\lambda)} \qquad W^\phi(q) = \lambda \tau' + (1-\lambda)$$

By Lemma 13, we have that $\Phi(p) = [\underline{\alpha}(p), \overline{\alpha}(p)] \times \tau(p)]$, which implies that $\tau' = \tau(p)$ and for some $\pi \in [0, 1]$ we have $\alpha' = \pi \overline{\alpha}(p) + (1-\pi)\underline{\alpha}(p)$, which implies that

$$U^\phi(q) = \frac{\lambda \tau(p)[\pi \overline{\alpha}(p) + (1-\pi)\underline{\alpha}(p)] + (1-\lambda)\theta}{\lambda \tau(p) + (1-\lambda)} = \pi \overline{U}(q) + (1-\pi)\underline{U}(q)$$

This implies that $U^\phi(q^1) > U^\phi(q^2)$ for every $(U^\phi, W^\phi) \in \mathcal{V}(p)$ and hence by Lemma

7, $q^1 \succ q^2$. ■

Proof of Lemma 16 Fix $p^1, p^2 \in \mathcal{C}$ and let

$$\lambda^* = \frac{1-\tau(p^1)}{1-\tau(p^2)} \qquad \theta^* = \frac{\tau(p^1)(1-\tau(p^2))\overline{\alpha}(p^1) - \tau(p^2)(1-\tau(p^1))\overline{\alpha}(p^2)}{\tau(p^1)-\tau(p^2)}$$

Then letting $q^2 = \lambda^* p^2 + (1-\lambda^*)\zeta_{\theta^*}$, we have that by Lemma 5,

$$\overline{\alpha}(q^2) = \frac{[1-\tau(p^1)]\tau(p^2)\overline{\alpha}(p^2) + [\tau(p^1)(1-\tau(p^2))\overline{\alpha}(p^1) - \tau(p^2)(1-\tau(p^1))\overline{\alpha}(p^2)]}{[1-\tau(p^1)]\tau(p^2) + [\tau(p^1)-\tau(p^2)]} = \overline{\alpha}(p^1)$$
$$\tau(q^2) = \frac{1-\tau(p^1)}{1-\tau(p^2)}\tau(p^2) + \frac{\tau(p^1)-\tau(p^2)}{1-\tau(p^2)} = \tau(p^1)$$

Fix $\pi \in [0,1]$ and let $q^\pi = \pi p^1 + (1-\pi)q^2$, and assume without loss of generality that $\tau(q^\pi) < \tau(p^1)$. Then for some $\delta \in \mathbb{R}$ sufficiently large we have that

$$\underline{\alpha}(q^\pi + \delta(\overline{p} - \underline{p})) = \underline{\alpha}(q^\pi) + \frac{\delta}{\tau(q^\pi)} > \overline{\alpha}(p^1) + \frac{\delta}{\tau(p^1)} = \overline{\alpha}(p^1 + \delta(\overline{p} - \underline{p})) = \overline{\alpha}(q^2 + \delta(\overline{p} - \underline{p}))$$

This implies that there is $\alpha' \in [0,1]$ such that $p^1 + \delta(\overline{p} - \underline{p}) \prec \zeta_{\alpha'}$ and $q^2 + \delta(\overline{p} - \underline{p}) \prec \zeta_{\alpha'}$ but $q^\pi + \delta(\overline{p} - \underline{p}) = \pi[p^1 + \delta(\overline{p} - \underline{p})] + (1-\pi)[q^2 + \delta(\overline{p} - \underline{p})] \succ \zeta_{\alpha'}$, violating mixture dominance. A similar argument shows that we cannot have $\tau(q^\pi) > \tau(p^1)$, and thus we must have that $\tau(q^\pi) = \tau(p^1)$. Thus letting $\pi' = \frac{\pi(1-\tau(p^2))}{\pi(1-\tau(p^2))+(1-\pi)(1-\tau(p^1))}$ and $\lambda' = \frac{\pi(1-\tau(p^2))+(1-\pi)(1-\tau(p^1))}{1-\tau(p^2)}$ we have that

$$\begin{aligned} q^\pi &= \pi p^1 + (1-\pi)\left[\frac{1-\tau(p^1)}{1-\tau(p^2)}p^2 + \frac{\tau(p^1)-\tau(p^2)}{1-\tau(p^2)}\zeta_{\frac{\tau(p^1)(1-\tau(p^2))\overline{\alpha}(p^1)-\tau(p^2)(1-\tau(p^1))\overline{\alpha}(p^2)}{\tau(p^1)-\tau(p^2)}}\right] \\ &= \lambda'[\pi' p^1 + (1-\pi')p^2] + (1-\lambda')\zeta_{\theta^*} \end{aligned}$$

Hence letting $p^{\pi'} = \pi' p^1 + (1-\pi')p^2$ we have by Lemma 5 we have that

$$\begin{aligned}
\tau(p^{\pi'}) &= \frac{1}{\lambda'}\tau(p^1) + \left(1 - \frac{1}{\lambda'}\right) = \frac{(1-\tau(p^2))\tau(p^1) + (1-\pi)(\tau(p^2) - \tau(p^1))}{\pi(1-\tau(p^2)) + (1-\pi)(1-\tau(p^1))} \\
&= \frac{\pi(1-\tau(p^2))\tau(p^1) + (1-\pi)(1-\tau(p^1))\tau(p^2)}{\pi(1-\tau(p^2)) + (1-\pi)(1-\tau(p^1))} \\
&= \pi'\tau(p^1) + (1-\pi')\tau(p^2)
\end{aligned}$$

Furthermore, since by mixture dominance we have that $\alpha' \succ p^1$ and $\alpha' \succ q^2$ implies $\alpha' \succ q^\pi$, this implies that $\overline{\alpha}(q^\pi) \leq \overline{\alpha}(p^1)$, so that again by Lemma 5

$$\begin{aligned}
\overline{\alpha}(p^{\pi'}) &= \frac{\frac{1}{\lambda'}\tau(p^1)\overline{\alpha}(q^\pi) + \left(1 - \frac{1}{\lambda'}\right)\theta^*}{\frac{1}{\lambda'}\tau(p^1) + \left(1 - \frac{1}{\lambda'}\right)} \\
&\leq \frac{(1-\tau(p^2))\tau(p^1)\overline{\alpha}(p^1) + (1-\pi)[\tau(p^2)(1-\tau(p^1))\overline{\alpha}(p^2) - \tau(p^1)(1-\tau(p^2))\overline{\alpha}(p^1)]}{(1-\tau(p^2))\tau(p^1) + (1-\pi)(\tau(p^2) - \tau(p^1))} \\
&= \frac{\pi(1-\tau(p^2))\tau(p^1)\overline{\alpha}(p^1) + (1-\pi)(1-\tau(p^1))\tau(p^2)\overline{\alpha}(p^2)}{\pi(1-\tau(p^2))\tau(p^1) + (1-\pi)(1-\tau(p^1))\tau(p^2)} \\
&= \frac{\pi'\tau(p^1)\overline{\alpha}(p^1) + (1-\pi')\tau(p^2)\overline{\alpha}(p^2)}{\pi'\tau(p^1) + (1-\pi')\tau(p^2)}
\end{aligned}$$

By a similar argument, we have that $\underline{\alpha}(q^\pi) \geq \underline{\alpha}(p^1)$ and hence

$$\underline{\alpha}(p^{\pi'}) \geq \frac{\pi'\tau(p^1)\underline{\alpha}(p^1) + (1-\pi')\tau(p^2)\underline{\alpha}(p^2)}{\pi'\tau(p^1) + (1-\pi')\tau(p^2)}$$

■

Proof of Theorem 3 We first establish the uniqueness of the weight function. By mixture dominance there are $\overline{x}, \underline{x} \in X$ such that $\delta_{\overline{x}} = \overline{p}$ and $\delta_{\underline{x}} = \underline{p}$. For every $x \in X$ and $u^i \in \tilde{\mathcal{U}}^i$ we have $u^i(\overline{x}) > u^i(x) > u^i(\underline{x})$, so that there is $\alpha \in (0,1)$ such that

$$u^i(x) = \frac{\alpha w^i(\overline{x})u^i(\overline{x}) + (1-\alpha)w^i(\underline{x})u^i(\underline{x})}{\alpha w^i(\overline{x}) + (1-\alpha)w^i(\underline{x})} = U^i(\zeta_\alpha)$$

By definition, this implies that $\delta_x \asymp \zeta_\alpha$ so that $\alpha \in A(\delta_x)$. For every $\beta \in (0,1)$, we have by weighted linearity that

$$U^i(\beta\delta_x + (1-\beta)\delta_{\overline{x}}) = \frac{\beta w^i(x)u^i(x) + (1-\beta)w^i(\overline{x})u^i(\overline{x})}{\beta w^i(x) + (1-\beta)w^i(\overline{x})}$$
$$= \frac{\beta w^i(x)\left[\frac{\alpha w^i(\overline{x})u^i(\overline{x}) + (1-\alpha)w^i(\underline{x})u^i(\underline{x})}{\alpha w^i(\overline{x}) + (1-\alpha)w^i(\underline{x})}\right] + (1-\beta)w^i(\overline{x})u^i(\overline{x})}{\beta w^i(x) + (1-\beta)w^i(\overline{x})}$$

On the other hand, by parallel substitution there is a unique τ_x such that for every $\beta \in (0,1)$, $\beta\delta_x + (1-\beta)\delta_{\overline{x}} \asymp \frac{\beta\tau_x\zeta_\alpha + (1-\beta)\delta_{\overline{x}}}{\beta\tau_x + (1-\beta)} = \zeta_{\frac{\beta\tau_x\alpha + (1-\beta)}{\beta\tau_x + (1-\beta)}}$, which implies that

$$U^i(\beta\delta_x + (1-\beta)\delta_{\overline{x}}) = \frac{[\beta\tau_x\alpha + (1-\beta)]w^i(\overline{x})u^i(\overline{x}) + \beta\tau_x(1-\alpha)w^i(\underline{x})u^i(\underline{x})}{[\beta\tau(\delta_x)\alpha + (1-\beta)]w^i(\overline{x}) + \beta\tau_x(1-\alpha)w^i(\underline{x})}$$
$$= \frac{\beta\tau_x[\alpha w^i(\overline{x})u^i(\overline{x}) + (1-\alpha)w^i(\underline{x})u^i(\underline{x})] + (1-\beta)w^i(\overline{x})u^i(\overline{x})}{\beta\tau_x[\alpha w^i(\overline{x}) + (1-\alpha)w^i(\underline{x})] + (1-\beta)w^i(\overline{x})}$$

This implies for $i = 1,2$ that $w^i(x) = \tau_x[\alpha w^i(\overline{x}) + (1-\alpha)w^i(\underline{x})]$ for every $\alpha \in A(\delta_x)$, and hence $w^i(\overline{x}) = w^i(\underline{x})$. Therefore letting $k = \frac{w^2(\overline{x})}{w^1(\overline{x})}$ we have that $w^2(x) = kw^1(x)$ for every $x \in X$.

We now establish the uniqueness of the convex cone generated by the set of utilities. We have for $p \in \Delta(X)$ and $x \in X$ that

$$p^{w^2}(x) = \frac{p(x)kw^1(x)}{\sum_{\xi\in X} p(\xi)kw^1(\xi)} = \frac{p(x)w^1(x)}{\sum_{\xi\in X} p(\xi)w^1(\xi)} = p^{w^1}(x)$$

Thus the transformed preference relations $\succ^{w^1}$ and $\succ^{w^2}$ are identical, denote it by $\succ^w$. Since $p \succ q$ if and only if $p^w \succ^w q^w$, the transformed relation $\succ^w$ has a multi-utility representation and hence satisfies independence. Define its domination cone by

$$\mathcal{D}^w = \{\lambda(p^w - q^w) : p^w \succ^w q^w, \lambda \geq 0\}$$

Note that $p^w - q^w \in \mathcal{D}^w$ if and only if $p^w \succ q^w$, since if $p^w - q^w = \lambda(r^w - s^w) \in \mathcal{D}^w$ for some $r^w, s^w \in \Delta(X)$ such that $r^w \succ^w s^w$ and $\lambda \geq 0$, by independence we have that

$$\frac{1}{1+\lambda}p^w + \frac{\lambda}{1+\lambda}s^w = \frac{1}{1+\lambda}q^w + \frac{\lambda}{1+\lambda}r^w \succ \frac{1}{1+\lambda}q^w + \frac{\lambda}{1+\lambda}s^w$$

This implies $p^w \succ q^w$ by another application of independence. Hence for $i = 1, 2$, $\rho \in \mathcal{D}^w$ if and only if $\sum_{x \in X} \rho(x)u^i(x) > 0$ for every $u^1 \in \langle\tilde{\mathcal{U}}^i\rangle$. Now suppose there is $u^* \in \langle\tilde{\mathcal{U}}^2\rangle \setminus \langle\tilde{\mathcal{U}}^1\rangle$. Then by the separating hyperplane theorem there is $\rho \in \mathbb{R}^{|X|}$ such that

$$\sum_{x \in X} \rho(x)u^1(x) > 0 \geq \sum_{x \in X} \rho(x)u^*(x) \quad \forall\, u^1 \in \langle\tilde{\mathcal{U}}^1\rangle$$

This implies that $\rho \in \mathcal{D}^w$, so that there are $p^w, q^w \in \Delta(X)$ and $\lambda \geq 0$ such that $\rho = \lambda(p^w - q^w)$, which implies that $\sum_{x \in X} p^w(x)u^1(x) > \sum_{x \in X} q^w(x)u^1(x)$ for every $u^1 \in \langle\tilde{\mathcal{U}}^1\rangle$ but $\sum_{x \in X} p^w(x)u^*(x) \leq \sum_{x \in X} q^w(x)u^*(x)$, a contradiction. Hence $\langle\tilde{\mathcal{U}}^2\rangle \setminus \langle\tilde{\mathcal{U}}^1\rangle = \varnothing$ and by a similar argument $\langle\tilde{\mathcal{U}}^1\rangle \setminus \langle\tilde{\mathcal{U}}^2\rangle = \varnothing$, so that we must have $\langle\tilde{\mathcal{U}}^1\rangle = \langle\tilde{\mathcal{U}}^2\rangle$. ■

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